

# TWO ERROR BOUNDS OF THE ELLIPTIC ASYMPTOTICS FOR THE FIFTH PAINLEVÉ TRANSCENDENTS

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ABSTRACT. For the fifth Painlevé equation it is known that a general solution is represented asymptotically by an elliptic function in cheese-like strips near the point at infinity. We present an explicit asymptotic formula for the error term of this expression, which leads to an estimate for its magnitude as was conjectured. An analogous formula is obtained for the error term of the correction function associated with the Lagrangian.

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## 1. INTRODUCTION

The fifth Painlevé equation

$$y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2} \left( a_\theta y - \frac{b_\theta}{y} \right) + c_\theta \frac{y}{x} - \frac{y(y+1)}{2(y-1)}, \quad (\text{P}_V)$$

in which  $8a_\theta = (\theta_0 - \theta_1 + \theta_\infty)^2$ ,  $8b_\theta = (\theta_0 - \theta_1 - \theta_\infty)^2$ ,  $c_\theta = 1 - \theta_0 - \theta_1$  with  $\theta_0, \theta_1, \theta_\infty \in \mathbb{C}$ , governs the isomonodromy deformation of a linear system of the form

$$\begin{aligned} \frac{d\xi}{d\lambda} &= \left( \frac{x}{2} \sigma_3 + \frac{\mathcal{A}_0}{\lambda} + \frac{\mathcal{A}_1}{\lambda-1} \right) \xi, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{A}_0 = \begin{pmatrix} \mathfrak{z} + \theta_0/2 & -u(\mathfrak{z} + \theta_0) \\ \mathfrak{z}/u & -\mathfrak{z} - \theta_0/2 \end{pmatrix}, \\ \mathcal{A}_1 &= \begin{pmatrix} -\mathfrak{z} - (\theta_0 + \theta_\infty)/2 & uy(\mathfrak{z} + (\theta_0 - \theta_1 + \theta_\infty)/2) \\ -(uy)^{-1}(\mathfrak{z} + (\theta_0 + \theta_1 + \theta_\infty)/2) & \mathfrak{z} + (\theta_0 + \theta_\infty)/2 \end{pmatrix} \end{aligned}$$

(cf. [1], [10, (1.1)], [11, (3.1)]) with the monodromy data  $(M^0, M^1) = ((m_{ij}^0), (m_{ij}^1)) \in SL_2(\mathbb{C})^2$  defined along loops surrounding  $\lambda = 0$  and  $1$ , respectively. Then a general solution  $y(x)$  of (P<sub>V</sub>) is parametrised by  $(M^0, M^1)$  [1, Section 2]. As in [10, 11, Theorem 2.1], for each  $\phi$  such that  $0 < |\phi| < \pi/2$ ,  $y(x)$  admits an expression of the form

$$\frac{y(x) + 1}{y(x) - 1} = A_\phi^{1/2} \operatorname{sn}((x - x_0)/2 + \Delta(x); A_\phi^{1/2}) \quad (1.1)$$

with  $\Delta(x) = O(x^{-2/9+\varepsilon})$  for any  $\varepsilon$  satisfying  $0 < \varepsilon < 2/9$  as  $x = e^{i\phi t} \rightarrow \infty$  through the cheese-like strip  $S(\phi, t_\infty, \kappa_0, \delta_0)$ , where  $v = \operatorname{sn}(z; k)$  is the Jacobi elliptic function such that  $v_z^2 = (1 - v^2)(1 - k^2 v^2)$ , and the symbols  $A_\phi$ ,  $x_0$  and  $S(\phi, t_\infty, \kappa_0, \delta_0)$  are as in **(1)** and **(3)** below. Since  $A_\phi$  does not depend on the solution  $y(x)$ , the leading term of the expression above contains the integration constant  $x_0$  depending on  $(M^0, M^1)$  and

the other integration constant appears in the error term  $\Delta(x)$ . Moreover  $\Delta(x)$  may be treated in studying, say, the  $\tau$ -function [8, p. 121], and degeneration into trigonometric asymptotics [8, Section 4]. For these facts detailed study on  $\Delta(x)$  is desirable. Under the supposition  $\Delta(x) = O(x^{-1})$ , an asymptotic form of  $\Delta(x)$  containing the other integration constant is discussed in [10, Theorem 2.3 and Corollary 2.4]. For the  $\tau$ -function associated with  $(P_I)$  Iwaki [5], by the method of topological recursion, obtained a conjectural full-order expansion yielding the elliptic expression of solutions.

In this paper we unconditionally present an explicit expression of  $\Delta(x)$ , which leads to the estimate  $\Delta(x) = O(x^{-1})$  as was conjectured. The correction function  $B_\phi(t)$  [11, (5.5)] for the Lagrangian of  $y(x)$  contains information about asymptotics (see also [8, Section 3]). An analogous explicit formula is obtained for the error term of the asymptotic expression of  $B_\phi(t)$ .

Our results are stated in Theorems 2.1, 2.2 and 2.3. In Section 3, from a system of equations equivalent to  $(P_V)$  we derive integral equations containing the error term  $h(x) = \Delta(x)/2$ . The final section is devoted to the proofs of main theorems by using these equations, in which our argument is quite different from those in [4], [2, Chapter 8], [6] and [9] applied to  $(P_{II})$  and  $(P_I)$ .

Throughout this paper we use the following symbols.

(1) For each  $\phi \in \mathbb{R}$ ,  $A_\phi \in \mathbb{C}$  is a unique solution of the Boutroux equations

$$\operatorname{Re} e^{i\phi} \int_{\mathbf{a}} \sqrt{\frac{A_\phi - z^2}{1 - z^2}} dz = \operatorname{Re} e^{i\phi} \int_{\mathbf{b}} \sqrt{\frac{A_\phi - z^2}{1 - z^2}} dz = 0$$

[10, Section 7]. Here  $\mathbf{a}$  and  $\mathbf{b}$  are basic cycles as in Figure 1.1 on the elliptic curve  $\Pi^* = \Pi_+^* \cup \Pi_-^*$  given by  $w(A_\phi, z) = \sqrt{(1 - z^2)(A_\phi - z^2)}$  such that  $\Pi_+^*$  and  $\Pi_-^*$  are glued along the cuts  $[-1, -A_\phi^{1/2}] \cup [A_\phi^{1/2}, 1]$  with  $0 \leq \operatorname{Re} A_\phi^{1/2} \leq 1$ ; and the branches of the square roots

$$\sqrt{\frac{A_\phi - z^2}{1 - z^2}} = \frac{\sqrt{A_\phi - z^2}}{\sqrt{1 - z^2}}, \quad \sqrt{(A_\phi - z^2)(1 - z^2)} = \sqrt{A_\phi - z^2} \sqrt{1 - z^2}$$

are determined by  $z^{-1} \sqrt{A_\phi - z^2} \rightarrow i$  and  $z^{-1} \sqrt{1 - z^2} \rightarrow i$  as  $z \rightarrow \infty$  on the upper sheet  $\Pi_+^*$ .

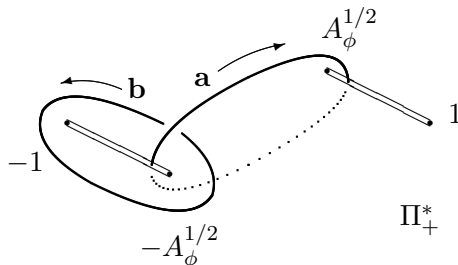


FIGURE 1.1. Cycles  $\mathbf{a}$ ,  $\mathbf{b}$  on  $\Pi^*$

(2) The periods of  $\Pi^*$  along  $\mathbf{a}$  and  $\mathbf{b}$  are

$$\Omega_{\mathbf{a}} = \int_{\mathbf{a}} \frac{dz}{w(A_\phi, z)}, \quad \Omega_{\mathbf{b}} = \int_{\mathbf{b}} \frac{dz}{w(A_\phi, z)},$$

and write

$$\mathcal{E}_{\mathbf{a}} = \int_{\mathbf{a}} \sqrt{\frac{A_\phi - z^2}{1 - z^2}} dz, \quad \mathcal{E}_{\mathbf{b}} = \int_{\mathbf{b}} \sqrt{\frac{A_\phi - z^2}{1 - z^2}} dz.$$

(3) Set

$$x_0 \equiv \frac{-1}{\pi i} (\Omega_{\mathbf{b}} \log(m_{21}^0 m_{12}^1) + \Omega_{\mathbf{a}} \log \mathbf{m}_\phi) - (\frac{1}{2}\Omega_{\mathbf{a}} + \Omega_{\mathbf{b}})(\theta_\infty + 1) - \frac{1}{2}\Omega_{\mathbf{a}} \pmod{2\Omega_{\mathbf{a}}\mathbb{Z} + 2\Omega_{\mathbf{b}}\mathbb{Z}},$$

in which  $\mathbf{m}_\phi = m_{11}^0$  if  $-\pi/2 < \phi < 0$ , and  $= e^{-\pi i \theta_\infty} (m_{11}^1)^{-1}$  if  $0 < \phi < \pi/2$ . For given positive numbers  $\kappa_0$ ,  $\delta_0$  and  $t_\infty$ ,

$$S(\phi, t_\infty, \kappa_0, \delta_0) = \{x = e^{i\phi} t \mid \operatorname{Re} t > t_\infty, \quad |\operatorname{Im} t| < \kappa_0\} \setminus \bigcup_{\sigma \in \mathcal{P}_0} \{|x - \sigma| < \delta_0\}$$

with  $\mathcal{P}_0 = \{\sigma \mid \operatorname{sn}((\sigma - x_0)/2; A_\phi^{1/2}) = \infty\} = \{x_0 + \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}(2\mathbb{Z} + 1)\}$ , and

$$\check{S}(\phi, t_\infty, \kappa_0, \delta_0) = S(\phi, t_\infty, \kappa_0, \delta_0) \setminus \bigcup_{\sigma \in \mathcal{Q}} \{|x - \sigma| < \delta_0\}$$

with  $\mathcal{Q} = \{\sigma \mid \operatorname{sn}((\sigma - x_0)/2; A_\phi^{1/2}) = \pm A_\phi^{-1/2}, \pm 1\}$ , in which  $\delta_0$  is also supposed so small that  $\{|x - \sigma_1| = \delta_0\} \cap \{|x - \sigma_2| = \delta_0\} = \emptyset$  for any  $\sigma_1, \sigma_2 \in \mathcal{P}_0 \cup \mathcal{Q}$ ,  $\sigma_1 \neq \sigma_2$ . For  $\sigma = e^{i\phi} t_\sigma \in \mathcal{Q}$  let  $l(\sigma)$  be the line defined by  $x = e^{i\phi}(\operatorname{Re} t_\sigma + i\eta)$  with  $\eta \geq \operatorname{Im} t_\sigma$  if  $\operatorname{Im} t_\sigma \geq 0$  (respectively,  $\eta \leq \operatorname{Im} t_\sigma$  if  $\operatorname{Im} t_\sigma < 0$ ); and, if necessary, modify  $l(\sigma)$  not to touch other circles  $\{|x - \sigma'| = \delta_0\}$  with  $\sigma' \in \mathcal{P}_0 \cup \mathcal{Q} \setminus \{\sigma\}$  by suitable replacement of local segments on  $l(\sigma)$  with arcs. Then let  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$  denote  $\check{S}(\phi, t_\infty, \kappa_0, \delta_0)$  equipped with the cuts along  $l(\sigma)$  or its modification for all  $\sigma \in \mathcal{Q}$  (cf. Figure 1.2).

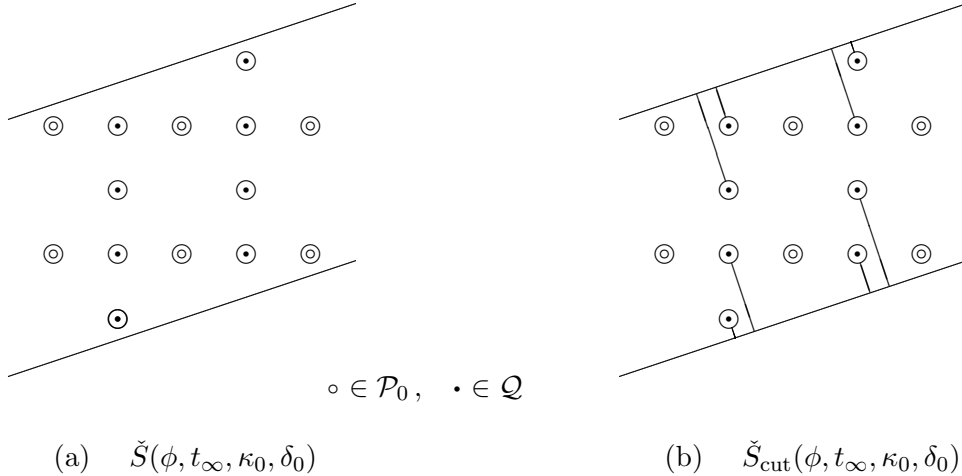


FIGURE 1.2. Cheese-like strips

(4) For  $\operatorname{Im} \tau > 0$ ,

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i z n}$$

with  $\vartheta'(z, \tau) = (d/dz)\vartheta(z, \tau)$  is the  $\vartheta$ -function [3], [12]. Note that  $\vartheta(z \pm 1, \tau) = \vartheta(z, \tau)$ ,  $\vartheta(z \pm \tau, \tau) = e^{-\pi i(\tau \pm 2z)} \vartheta(z, \tau)$ .

(5) We write  $f \ll g$  or  $g \gg f$  if  $f = O(g)$ .

## 2. MAIN RESULTS

Our results are stated as follows.

**Theorem 2.1.** *Suppose that  $0 < |\phi| < \pi/2$ . Let  $y(x)$  be the solution of  $(P_V)$  corresponding to  $(M^0, M^1) = ((m_{ij}^0), (m_{ij}^1))$  with  $m_{11}^0 m_{11}^1 m_{21}^0 m_{12}^1 \neq 0$ . Then*

$$\frac{y(x) + 1}{y(x) - 1} = A_\phi^{1/2} \operatorname{sn}((x - x_0)/2; A_\phi^{1/2}) + O(x^{-1})$$

as  $x \rightarrow \infty$  through the cheese-like strip  $S(\phi, t_\infty, \kappa_0, \delta_0)$ , where  $\kappa_0$  is a given number,  $\delta_0$  a given small number, and  $t_\infty = t_\infty(\kappa_0, \delta_0)$  a large number depending on  $(\kappa_0, \delta_0)$ .

Set

$$\begin{aligned} \psi_0(x) &= A_\phi^{1/2} \operatorname{sn}((x - x_0)/2; A_\phi^{1/2}), \\ b_0(x) &= \beta_0 - \frac{2\mathcal{E}_a}{\Omega_a} x - \frac{8}{\Omega_a} \frac{\vartheta'}{\vartheta} \left( \frac{1}{2\Omega_a} (x - x_0), \tau_0 \right), \quad \tau_0 = \frac{\Omega_b}{\Omega_a}, \\ \mathbf{b}(x) &= \frac{\mathcal{E}_a}{4} (x - x_0) + \frac{\vartheta'}{\vartheta} \left( \frac{1}{2\Omega_a} (x - x_0), \tau_0 \right) = -\frac{\Omega_a}{8} (b_0(x) - b_0(x_0)) \end{aligned}$$

with

$$\beta_0 = -\frac{8}{\Omega_a} (\log(m_{21}^0 m_{12}^1) + \pi i(\theta_\infty + 1)), \quad b_0(x_0) = \beta_0 - \frac{2\mathcal{E}_a}{\Omega_a} x_0,$$

where  $\psi_0(x)$ ,  $b_0(x)$  and  $\mathbf{b}(x)$  are bounded in  $S(\phi, t_\infty, \kappa_0, \delta_0)$ .

**Theorem 2.2.** *The error term  $\Delta(x) = h(x)/2$  in (1.1) is represented by*

$$h(x) = -\frac{2((\theta_0 - \theta_1)^2 + \theta_\infty^2)}{A_\phi - 1} x^{-1} - \int_\infty^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} - \frac{3}{2} \int_\infty^x F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} + O(x^{-2}),$$

with

$$F_1(\psi_0, b_0) = \frac{4(\theta_0 + \theta_1)\psi_0 - b_0}{2(A_\phi - \psi_0^2)}, \quad \psi_0 = \psi_0(\xi), \quad b_0 = b_0(\xi).$$

Here

$$\int_\infty^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} \ll x^{-1}, \quad \int_\infty^x F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} \ll x^{-1}$$

as  $x \rightarrow \infty$  through  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ . Furthermore,

$$xh(x) = h_0 \beta_0^2 + h_1(x) \beta_0 + h_2(x) + O(x^{-1}),$$

where  $h_0 = (1/8)A_\phi^{-1}(1 - A_\phi)^{-1}$ ,  $h_1(x) \ll 1$ ,  $h_2(x) \ll 1$ .

*Remark 2.1.* Since  $(y(x) + 1)(y(x) - 1)^{-1} = \psi_0(x + h(x))$ , we have, in each neighbourhood of  $\sigma \in \check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ ,

$$\frac{y(x) + 1}{y(x) - 1} - A_\phi^{1/2} \operatorname{sn}((x - x_0)/2; A_\phi^{1/2}) = \sum_{j=1}^{\infty} \frac{\psi_0^{(j)}(x)}{j!} h(x)^j \sim \psi_0'(x) h(x),$$

which implies the single-valuedness of  $h(x)$  in  $\check{S}(\phi, t_\infty, \kappa_0, \delta_0)$ . In showing Theorem 2.2, for convenience' sake, the integral representations have been treated in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ , in which a contour joining  $x$  to  $\infty$  is topologically specified, to avoid the possible multi-valuedness of each integral around the pole at  $\sigma \in \mathcal{Q}$ .

*Remark 2.2.* Note that  $(P_V)$  is equivalent to the system

$$\begin{aligned}\mathfrak{z} &= \frac{x(y-y')}{2(y-1)^2} + \frac{\theta_0 + \theta_1}{2(y-1)} - \frac{1}{4}(\theta_0 - \theta_1 + \theta_\infty), \\ x\mathfrak{z}' &= y\mathfrak{z}(\mathfrak{z} + \frac{1}{2}(\theta_0 - \theta_1 + \theta_\infty)) - y^{-1}(\mathfrak{z} + \theta_0)(\mathfrak{z} + \frac{1}{2}(\theta_0 + \theta_1 + \theta_\infty))\end{aligned}$$

governing the isomonodromy deformation [1]. By Theorem 2.2, from (1.1) it follows that

$$-\frac{y'(x)}{(y(x)-1)^2} = \frac{A_\phi^{1/2}}{8} \left( 2 \operatorname{sn}'(\frac{1}{2}(x-x_0))(1+h'(x)) + \operatorname{sn}''(\frac{1}{2}(x-x_0))h(x) \right) + O(x^{-2})$$

and  $2(y-1)^{-1} = A_\phi^{1/2}(\operatorname{sn}(\frac{1}{2}(x-x_0)) + \frac{1}{2}\operatorname{sn}'(\frac{1}{2}(x-x_0))h(x)) - 1$  in  $S(\phi, t_\infty, \kappa_0, \delta_0)$ , where  $h(x)$ ,  $h'(x) \ll x^{-1}$ ,  $\operatorname{sn} z = \operatorname{sn}(z, A_\phi^{1/2})$  and  $\operatorname{sn}' z = \frac{d}{dz} \operatorname{sn} z$ . Then we have

$$\begin{aligned}\mathfrak{z}(x) &= \frac{x}{8} (A_\phi^{1/2} \operatorname{sn}'(\frac{1}{2}(x-x_0)) + A_\phi \operatorname{sn}^2(\frac{1}{2}(x-x_0)) - 1) \\ &\quad + \frac{xh(x)}{16} (A_\phi^{1/2} \operatorname{sn}''(\frac{1}{2}(x-x_0)) + 2A_\phi \operatorname{sn}(\frac{1}{2}(x-x_0)) \operatorname{sn}'(\frac{1}{2}(x-x_0))) \\ &\quad + \frac{xh'(x)}{8} A_\phi^{1/2} \operatorname{sn}'(\frac{1}{2}(x-x_0)) + \frac{\theta_0 + \theta_1}{4} A_\phi^{1/2} \operatorname{sn}(\frac{1}{2}(x-x_0)) - \frac{2\theta_0 + \theta_\infty}{4} + O(x^{-1}).\end{aligned}$$

Recall the correction function  $B_\phi(t)$  such that  $a_\phi = A_\phi + t^{-1}B_\phi(t)$ , where

$$\begin{aligned}a_\phi &= 1 - \frac{4(e^{-2i\phi}(y^*)^2 - y^2)}{y(y-1)^2} + 4e^{-i\phi}(\theta_0 + \theta_1) \frac{y+1}{y-1} t^{-1} \\ &\quad + e^{-2i\phi} \frac{(y-1)}{y} ((\theta_0 - \theta_1 + \theta_\infty)^2 y - (\theta_0 - \theta_1 - \theta_\infty)^2) t^{-2}\end{aligned}$$

with  $x = e^{i\phi t}$  [11, (3.5)]. In particular,  $a_{\phi, \text{Lag}} := a_\phi|_{y^*=dy/dt}$  is the Lagrangian of  $y = y(e^{i\phi t})$ . In this case, let  $b(x)$  be such that

$$a_{\phi, \text{Lag}} = A_\phi + \frac{b(x)}{x} = A_\phi + \frac{e^{-i\phi} b(e^{i\phi t})}{t}.$$

If  $y^* = dy/dt$ , then  $b(x) = e^{i\phi} B_\phi(t)$ . (In [10],  $a_\phi$  and  $B_\phi(t)$  are defined under the condition  $y^* = dy/dt$ .)

**Theorem 2.3.** *Under the same suppositions as in Theorems 2.1 and 2.2,*

$$\begin{aligned}b(x) - b_0(x) &= b'_0(x)h(x) - 4((\theta_0 - \theta_1)^2 + \theta_\infty^2)x^{-1} \\ &\quad - \int_\infty^x (A_\phi - \psi_0^2) F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} + O(x^{-2}),\end{aligned}$$

in which  $b'_0(x) = 4\psi'_0 - 2(A_\phi - \psi_0^2)$ , and

$$\int_\infty^x (A_\phi - \psi_0^2) F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} \ll x^{-1}$$

as  $x \rightarrow \infty$  through  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ , and  $b(x) - b_0(x) \ll x^{-1}$  in  $S(\phi, t_\infty, \kappa_0, \delta_0)$ .

*Remark 2.3.* As calculated in Section 4 the integrals above are written in the form

$$\begin{aligned} \int_{\infty}^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} &= 2(\theta_0 + \theta_1) \int_{\infty}^x \frac{\psi_0}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} \\ &\quad - \frac{b_0(x_0)}{2} \int_{\infty}^x \frac{1}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} + \frac{4}{\Omega_{\mathbf{a}}} \int_{\infty}^x \frac{\mathbf{b}}{A_\phi - \psi_0^2} \frac{d\xi}{\xi}, \\ \int_{\infty}^x F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} &= \frac{16(\theta_0 + \theta_1)^2 A_\phi + b_0(x_0)^2}{12A_\phi(A_\phi - 1)} x^{-1} + \frac{4b_0(x_0)}{3A_\phi(A_\phi - 1)\Omega_{\mathbf{a}}} \int_{\infty}^x \frac{\mathbf{b}}{\xi^2} d\xi \\ &\quad + \frac{16(\theta_0 + \theta_1)}{\Omega_{\mathbf{a}}} \int_{\infty}^x \frac{\mathbf{b}\psi_0}{(A_\phi - \psi_0^2)^2} \frac{d\xi}{\xi^2} + \frac{16}{\Omega_{\mathbf{a}}^2} \int_{\infty}^x \frac{\mathbf{b}^2}{(A_\phi - \psi_0^2)^2} \frac{d\xi}{\xi^2} + O(x^{-2}), \\ \int_{\infty}^x (A_\phi - \psi_0^2) F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} &= 4(\theta_0 + \theta_1)^2 x^{-1} \\ &\quad + \frac{16(\theta_0 + \theta_1)}{\Omega_{\mathbf{a}}} \int_{\infty}^x \frac{\mathbf{b}\psi_0}{A_\phi - \psi_0^2} \frac{d\xi}{\xi^2} + \frac{16}{\Omega_{\mathbf{a}}^2} \int_{\infty}^x \frac{\mathbf{b}^2}{A_\phi - \psi_0^2} \frac{d\xi}{\xi^2} + O(x^{-2}), \end{aligned}$$

in which each integral on the right-hand sides is  $O(x^{-1})$ , and immediately yield detailed expressions of  $h(x)$  and  $b(x) - b_0(x)$  (see Section 4.3).

### 3. SYSTEM OF INTEGRAL EQUATIONS

To prove our theorems, we recall the following facts [10, Section 6].

(1) For the solution  $y(x)$  of (P<sub>V</sub>),  $(\psi(x), b(x))$  with  $\psi(x) = (y(x) + 1)(y(x) - 1)^{-1}$  solves a system of equations

$$\begin{aligned} 4(\psi')^2 &= (1 - \psi^2)(A_\phi - \psi^2) - (1 - \psi^2)(4(\theta_0 + \theta_1)\psi - b)x^{-1} \\ &\quad + 4(2(\theta_0 - \theta_1)\theta_\infty\psi + (\theta_0 - \theta_1)^2 + \theta_\infty^2)x^{-2}, \end{aligned} \quad (3.1)$$

$$b' = -2(A_\phi - \psi^2) + 4\psi' + (4(\theta_0 + \theta_1)\psi - b)x^{-1}, \quad (3.2)$$

where  $b = b(x)$  is as defined in Section 2 by using the Lagrangian  $a_{\phi, \text{Lag}}$ .

(2)  $\psi_0(x)$  and  $b_0(x)$  are bounded in  $S(\phi, t_\infty, \kappa_0, \delta_0)$  and fulfil

$$4(\psi_0')^2 = (1 - \psi_0^2)(A_\phi - \psi_0^2), \quad (3.3)$$

$$b_0' = -2(A_\phi - \psi_0^2) + 4\psi_0' \quad (3.4)$$

[10], which at least formally approximates system (3.1), (3.2).

**Proposition 3.1.** *Equations (3.1), (3.2) admit a solution  $(\psi(x), b(x))$  such that  $\psi(x) = \psi_0(x + h(x))$  with  $h(x) \ll x^{-2/9+\varepsilon}$  and  $b(x) - b_0(x) \ll x^{-2/9+\varepsilon}$  as  $x \rightarrow \infty$  through  $S(\phi, t_\infty, \kappa_0, \delta_0)$  for any  $0 < \varepsilon < 2/9$ . Furthermore  $b_0(x)$  and  $b(x)$  are bounded.*

*Proof.* As in [11, Section 5], the correction function  $b^*(x) := e^{i\phi} B_\phi(t)$  admits the asymptotic expression  $b^*(x) - b_0(x) \ll x^{-2/9+\varepsilon}$ , which follows from [11, (5.5)] with  $\delta = 2/9 - \varepsilon$ . (Here we note that, in the argument of [11, Sectios 4 and 5] as well,  $\delta$  is so chosen in accordance with the annulus  $\mathcal{A}_\varepsilon$  in [10, p. 64].) By the justification scheme [11, Section 5] with [7], for  $b(x)$  corresponding to the Lagrangian  $a_{\phi, \text{Lag}}$  as well, the estimate  $b(x) - b_0(x) \ll x^{-2/9+\varepsilon}$  remains valid.  $\square$

From Proposition 3.1 with (3.3), it follows that  $2\psi'(x) = 2(1 + h'(x))\psi'_0(x + h(x)) = (1 + h')\sqrt{(1 - \psi_0(x + h)^2)(A_\phi - \psi_0(x + h)^2)} = (1 + h')\sqrt{(1 - \psi^2)(A_\phi - \psi^2)}$ . Then (3.1) becomes  $(1 + h')^2 = 1 - 2F_1(\psi, b)x^{-1} + 2F_2(\psi)x^{-2}$ , which yields

$$h' = -F_1(\psi, b)x^{-1} + (F_2(\psi) - \frac{1}{2}F_1(\psi, b)^2)x^{-2} + O(x^{-3}) \quad (3.5)$$

in  $\check{S}(\phi, t_\infty, \kappa_0, \delta_0)$ , where

$$F_1(\psi, b) = \frac{4(\theta_0 + \theta_1)\psi - b}{2(A_\phi - \psi^2)}, \quad F_2(\psi) = \frac{2(2(\theta_0 - \theta_1)\theta_\infty\psi + (\theta_0 - \theta_1)^2 + \theta_\infty^2)}{(1 - \psi^2)(A_\phi - \psi^2)}.$$

Using  $\psi = \psi_0 + \psi'_0 h + O(h^2)$ , we have

$$\begin{aligned} h' &= -F_1(\psi_0, b)x^{-1} + (F_2(\psi_0) - \frac{1}{2}F_1(\psi_0, b)^2)x^{-2} \\ &\quad - (F_1)_\psi(\psi_0, b)\psi'_0 h x^{-1} + O(x^{-1}(|x^{-1}| + |h|)^2). \end{aligned} \quad (3.6)$$

In what follows we suppose that, for a positive number  $\mu \leq 1$ ,

$$h(x) \ll x^{-\mu} \quad (3.7)$$

in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ . By Proposition 3.1, estimate (3.7) is true if, say,  $\mu = 1/9$ .

Let  $\{x_\nu\} \subset \check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$  be a given sequence such that  $|x_1| < \dots < |x_\nu| < \dots$ ,  $|x_\nu| \rightarrow \infty$ . Then, by (3.2) and (3.4)

$$\begin{aligned} b(x) - b(x_\nu) &= \int_{x_\nu}^x (4\psi' - 2(A_\phi - \psi^2))d\xi + \int_{x_\nu}^x (4(\theta_0 + \theta_1)\psi - b)\frac{d\xi}{\xi}, \\ b_0(x) - b_0(x_\nu) &= \int_{x_\nu}^x (4\psi'_0 - 2(A_\phi - \psi_0^2))d\xi, \end{aligned}$$

from which we derive, for  $x \in \check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$  with  $|x| < |x_\nu|$ ,

$$\begin{aligned} b(x) - b_0(x) - (b(x_\nu) - b_0(x_\nu)) &= 4(\psi(x) - \psi_0(x) - (\psi(x_\nu) - \psi_0(x_\nu))) \\ &\quad + 2 \int_{x_\nu}^x (\psi^2 - \psi_0^2)d\xi + 2 \int_{x_\nu}^x (A_\phi - \psi^2)F_1(\psi, b)\frac{d\xi}{\xi}. \end{aligned} \quad (3.8)$$

In this equality, by (3.7) and Proposition 3.1,

$$\begin{aligned} \psi(x) - \psi_0(x) - (\psi(x_\nu) - \psi_0(x_\nu)) &\ll |h(x)| + |h(x_\nu)| \ll |x^{-\mu}| + |x_\nu^{-\mu}|, \\ b(x_\nu) - b_0(x_\nu) &\ll x_\nu^{-2/9+\varepsilon}. \end{aligned}$$

Furthermore,

$$\begin{aligned} 2 \int_{x_\nu}^x (\psi^2 - \psi_0^2)d\xi &= 2 \int_{x_\nu}^x \left( (\psi_0^2)'h + \frac{(\psi_0^2)''}{2}h^2 + \dots + \frac{(\psi_0^2)^{(p)}}{p!}h^p + O(h^{p+1}) \right) d\xi \\ &= -2 \int_{x_\nu}^x \left( \psi_0^2 + (\psi_0^2)'h + \dots + \frac{(\psi_0^2)^{(p-1)}}{(p-1)!}h^{p-1} \right) h' d\xi + O(|x^{-\mu}| + |x_\nu^{-\mu}|), \end{aligned}$$

if  $-(p+1)\mu + 1 \leq -\mu$ , i.e.  $-p\mu + 1 \leq 0$ ; and by (3.5) and (3.7),

$$\begin{aligned} 2 \int_{x_\nu}^x (A_\phi - \psi^2)F_1(\psi, b)\frac{d\xi}{\xi} &= -2 \int_{x_\nu}^x (A_\phi - \psi^2)(h' + O(\xi^{-2}))d\xi \\ &= -2 \int_{x_\nu}^x (A_\phi - \psi^2)h'd\xi + O(|x^{-1}| + |x_\nu^{-1}|), \end{aligned}$$

where

$$\begin{aligned} & -2 \int_{x_\nu}^x (A_\phi - \psi^2) h' d\xi \\ & = 2 \int_{x_\nu}^x \left( \psi_0^2 + (\psi_0^2)' h + \dots + \frac{(\psi_0^2)^{(p-1)}}{(p-1)!} h^{p-1} \right) h' d\xi + O(|x^{-\mu}| + |x_\nu^{-\mu}|), \end{aligned}$$

since  $h' \ll \xi^{-1}$  by (3.6) and (3.7). Insert these quantities with  $p$  such that  $-p\mu + 1 \leq 0$  into (3.8). Under the passage to the limit  $x_\nu \rightarrow \infty$ , we arrive at the estimate

$$b(x) - b_0(x) \ll x^{-\mu} \quad (3.9)$$

in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ . Then equation (3.6) is written in the form

$$\begin{aligned} h' & = -F_1(\psi_0, b)x^{-1} + (F_2(\psi_0) - \frac{1}{2}F_1(\psi_0, b_0)^2)x^{-2} - (F_1)_\psi(\psi_0, b_0)\psi_0' h x^{-1} + O(x^{-1-2\mu}) \\ & = -F_1(\psi_0, b_0)x^{-1} + O(x^{-1-\mu}). \end{aligned} \quad (3.10)$$

For any sequence  $\{x_\nu\} \subset \check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ , integration of this yields

$$\begin{aligned} & h(x) - h(x_\nu) \\ & = - \int_{x_\nu}^x F_1(\psi_0, b) \frac{d\xi}{\xi} + \int_{x_\nu}^x (F_2(\psi_0) - \frac{1}{2}F_1(\psi_0, b_0)^2) \frac{d\xi}{\xi^2} - \mathcal{I}_0 + O(|x^{-2\mu}| + |x_\nu^{-2\mu}|) \end{aligned}$$

with

$$\begin{aligned} \mathcal{I}_0 & = \int_{x_\nu}^x (F_1)_\psi(\psi_0, b_0)\psi_0' h \frac{d\xi}{\xi} \\ & = \int_{x_\nu}^x \left( F_1(\psi_0, 0)_\xi - \left( \frac{1}{2(A_\phi - \psi_0^2)} \right)_\xi b_0 \right) h \frac{d\xi}{\xi} \\ & = \int_{x_\nu}^x \left( F_1(\psi_0, 0)F_1(\psi_0, b_0) - \frac{b_0 F_1(\psi_0, b_0) - b_0' h \xi}{2(A_\phi - \psi_0^2)} \right) \frac{d\xi}{\xi^2} + O(|x^{-1-\mu}| + |x_\nu^{-1-\mu}|) \\ & = \int_{x_\nu}^x F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} + \frac{1}{2} \int_{x_\nu}^x \frac{b_0' h}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} + O(|x^{-1-\mu}| + |x_\nu^{-1-\mu}|), \end{aligned}$$

where the third line is due to integration by parts. Hence we have, in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ ,

$$\begin{aligned} h(x) & = - \int_\infty^x F_1(\psi_0, b) \frac{d\xi}{\xi} \\ & \quad + \int_\infty^x \left( F_2(\psi_0) - \frac{3}{2}F_1(\psi_0, b_0)^2 \right) \frac{d\xi}{\xi^2} - \frac{1}{2} \int_\infty^x \frac{b_0' h}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} + O(x^{-2\mu}), \end{aligned} \quad (3.11)$$

in which the convergence of  $\int_\infty^x F_1(\psi_0, b)\xi^{-1}d\xi$  is guaranteed by the absolute convergence of the remaining two integrals.

By (3.1) and (3.2),

$$b' = -(A_\phi - \psi^2) + 4\psi' - \frac{4(\psi')^2}{1 - \psi^2} + 2(A_\phi - \psi^2)F_2(\psi)x^{-2}.$$

From this combined with (3.4) and  $2\psi' = (1 + h')\sqrt{(1 - \psi^2)(A_\phi - \psi^2)}$ , it follows that

$$(b - b_0)' = 2(\psi^2 - \psi_0^2) + 4(\psi - \psi_0)' - 2h'(A_\phi - \psi^2)(1 + h'/2) + 2(A_\phi - \psi^2)F_2(\psi)x^{-2},$$

and then, for any  $\{x_\nu\} \subset \check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ ,  $\chi := b - b_0$  satisfies

$$\begin{aligned} \chi(x) - \chi(x_\nu) &= 4(\psi - \psi_0) - 4(\psi(x_\nu) - \psi_0(x_\nu)) + 2 \int_{x_\nu}^x (\psi^2 - \psi_0^2 + h'\psi^2) d\xi \\ &\quad - 2A_\phi(h(x) - h(x_\nu)) - \int_{x_\nu}^x (A_\phi - \psi^2)((h')^2 - 2F_2(\psi)\xi^{-2}) d\xi. \end{aligned}$$

Observing that

$$\begin{aligned} &2 \int_{x_\nu}^x (\psi^2 - \psi_0^2 + h'\psi^2) d\xi \\ &= 2 \int_{x_\nu}^x \left( (\psi_0^2)'h + \dots + \frac{(\psi_0^2)^{(p)}}{p!} h^p \right. \\ &\quad \left. + h' \left( \psi_0^2 + \dots + \frac{(\psi_0^2)^{(p-1)}}{(p-1)!} h^{p-1} \right) + O(|h^{p+1}| + |h^p h'|) \right) d\xi \\ &= 2 \int_{x_\nu}^x \left( (\psi_0^2 h + \dots + \frac{(\psi_0^2)^{(p-1)}}{p!} h^p) \right)_\xi + O(|\xi^{-\mu(p+1)}| + |\xi^{-\mu p-1}|) d\xi \\ &= 2\psi_0^2 h + O(|h(x_\nu)| + |x^{-2\mu}| + |x_\nu^{-2\mu}|), \end{aligned}$$

if  $\mu(p-1) \geq 1$ , and that  $\psi(x_\nu) - \psi_0(x_\nu) \ll h(x_\nu)\psi_0'(x_\nu)$ , and using (3.10) and (3.4), we have

$$\begin{aligned} \chi &= (4\psi_0' - 2(A_\phi - \psi_0^2))h + \int_\infty^x (A_\phi - \psi_0^2)(2F_2(\psi_0) - F_1(\psi_0, b_0)^2) \frac{d\xi}{\xi^2} + O(x^{-2\mu}) \\ &= b_0' h + \int_\infty^x (A_\phi - \psi_0^2)(2F_2(\psi_0) - F_1(\psi_0, b_0)^2) \frac{d\xi}{\xi^2} + O(x^{-2\mu}). \end{aligned}$$

Combining this with (3.9) and (3.11) we have the following.

**Proposition 3.2.** *Under supposition (3.7) with  $0 < \mu \leq 1$ ,  $h$  and  $\chi = b - b_0$  satisfy*

$$\begin{aligned} h &= - \int_\infty^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} \\ &\quad + \int_\infty^x \left( F_2(\psi_0) - \frac{3}{2} F_1(\psi_0, b_0)^2 \right) \frac{d\xi}{\xi^2} + \frac{1}{2} \int_\infty^x \frac{\chi - b_0' h}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} + O(x^{-2\mu}), \\ \chi - b_0' h &= \int_\infty^x (A_\phi - \psi_0^2)(2F_2(\psi_0) - F_1(\psi_0, b_0)^2) \frac{d\xi}{\xi^2} + O(x^{-2\mu}) \end{aligned}$$

and  $\chi \ll x^{-\mu}$  in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ , in which each integral converges.

#### 4. PROOFS OF THE MAIN THEOREMS

Theorems 2.1 and 2.2 are immediately derived from the following proposition.

**Proposition 4.1.** *Under supposition (3.7) with  $0 < \mu \leq 1$ ,*

$$h(x) = - \frac{2((\theta_0 - \theta_1)^2 + \theta_\infty^2)}{A_\phi - 1} x^{-1} - \int_\infty^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} - \frac{3}{2} \int_\infty^x F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} + O(x^{-2\mu})$$

in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ , where

$$\int_\infty^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} \ll x^{-1}, \quad \int_\infty^x F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} \ll x^{-1}.$$

**Derivation of Theorems 2.1 and 2.2.** By Proposition 3.1 or [11, Theorem 2.1], estimate (3.7) with  $\mu = 1/9$  is valid, and Proposition 4.1 with  $\mu = 1/9$  leads us to (3.7) with  $h(x) \ll x^{-2/9}$  in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ . Then Proposition 4.1 with  $\mu = 2/9$  yields the asymptotic formula for  $h(x)$  with the error term  $O(x^{-4/9})$  and the estimate  $h(x) \ll x^{-4/9}$ . Twice more repetition of this procedure leads us to the desired asymptotic formula for  $h(x)$  of Theorem 2.2 in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ . By Remark 2.1, in  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ ,

$$(y(x) + 1)(y(x) - 1)^{-1} - A_\phi^{1/2} \text{sn}((x - x_0)/2; A_\phi^{1/2}) \ll \psi'_0(x)h(x),$$

where the left-hand side is holomorphic in  $S(\phi_0, t_\infty, \kappa_0, \delta_0)$ . By the maximal modulus principle, we have Theorem 2.1.  $\square$

*Remark 4.1.* By the argument above with (3.9) or Proposition 3.2, in  $S(\phi, t_\infty, \kappa_0, \delta_0)$

$$b(x) = b_0(x) + O(x^{-1}).$$

To complete the proofs of Theorems 2.1 and 2.2 it remains to establish Proposition 4.1. The main part of the proof consists of evaluation of integrals, in which the following primitive functions are used [10, Lemma 6.3].

**Lemma 4.2.** *Let  $\nu_0 = (1 + \tau_0)/2$  with  $\tau_0 = \Omega_{\mathbf{b}}/\Omega_{\mathbf{a}}$ . Then, for  $\text{sn } u = \text{sn}(u; A_\phi^{1/2})$ ,*

$$\begin{aligned} \int_0^u \frac{du}{1 - \text{sn}^2 u} &= \frac{1}{(A_\phi - 1)\Omega_{\mathbf{a}}} \\ &\quad \times \left( \mathcal{E}_{\mathbf{a}} u + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4} + \nu_0, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4} + \nu_0, \tau_0 \right) + c_1 \right), \\ \int_0^u \frac{\text{sn } u \, du}{1 - \text{sn}^2 u} &= \frac{1}{(A_\phi - 1)\Omega_{\mathbf{a}}} \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4} + \nu_0, \tau_0 \right) - \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4} + \nu_0, \tau_0 \right) + c_2 \right), \\ \int_0^u \frac{du}{1 - A_\phi \text{sn}^2 u} &= \frac{1}{(1 - A_\phi)\Omega_{\mathbf{a}}} \left( \mathcal{E}_{\mathbf{a}} u + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4}, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4}, \tau_0 \right) \right) + u, \\ \int_0^u \frac{\text{sn } u \, du}{1 - A_\phi \text{sn}^2 u} &= \frac{1}{A_\phi^{1/2}(1 - A_\phi)\Omega_{\mathbf{a}}} \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4}, \tau_0 \right) - \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4}, \tau_0 \right) + c_3 \right), \\ \int_0^u \frac{du}{(1 - \text{sn}^2 u)^2} &= \frac{-1}{6(A_\phi - 1)^2 \Omega_{\mathbf{a}}} \left( \left( \frac{d}{du} \right)^2 + 4(1 - 2A_\phi) \right) \left( \mathcal{E}_{\mathbf{a}} u + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4} + \nu_0, \tau_0 \right) \right. \\ &\quad \left. + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4} + \nu_0, \tau_0 \right) \right) - \frac{A_\phi}{3(A_\phi - 1)} u, \\ \int_0^u \frac{\text{sn } u \, du}{(1 - \text{sn}^2 u)^2} &= \frac{-1}{6(A_\phi - 1)^2 \Omega_{\mathbf{a}}} \left( \left( \frac{d}{du} \right)^2 + 1 - 5A_\phi \right) \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4} + \nu_0, \tau_0 \right) \right. \\ &\quad \left. - \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4} + \nu_0, \tau_0 \right) + c_4 \right), \end{aligned}$$

where  $c_j$  ( $1 \leq j \leq 4$ ) are some constants.

*Proof.* Recall the notation  $4K = \Omega_{\mathbf{a}}$ ,  $2iK' = \Omega_{\mathbf{b}}$  and  $k = A_\phi^{1/2}$ . Observing the behaviours around the poles  $u = 2K \pm K$ , we have

$$(k^2 - 1)(\operatorname{sn}^2 u - 1)^{-1} \equiv k^2(\operatorname{sn}^2(u - K + iK') - 1),$$

and

$$k^2(\operatorname{sn}^2(u - K + iK') - 1) + \frac{1}{\Omega_{\mathbf{a}}} \frac{d}{du} \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4} + \nu_0, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4} + \nu_0, \tau_0 \right) \right) \equiv c_0$$

(cf. [3], [12]). Integration on  $[-iK', -iK' + K]$  yields  $c_0 = -\mathcal{E}_{\mathbf{a}}/\Omega_{\mathbf{a}}$ , which implies the first formula. From

$$(k^2 - 1)^2(\operatorname{sn}^2 u - 1)^{-2} \equiv k^4 \operatorname{cn}^4(u - K + iK'),$$

and

$$k^4 \operatorname{cn}^4(u - K + iK') + \left( \frac{1}{6\Omega_{\mathbf{a}}} \left( \frac{d}{du} \right)^3 + \frac{2(1 - 2k^2)}{3\Omega_{\mathbf{a}}} \frac{d}{du} \right) \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} - \frac{1}{4} + \nu_0, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_{\mathbf{a}}} + \frac{1}{4} + \nu_0, \tau_0 \right) \right) \equiv c_0$$

with

$$k^4 \int_0^K \operatorname{cn}^4 u du = \frac{1}{6}(2k^2 - 1)\mathcal{E}_{\mathbf{a}} + \frac{k^2}{12}(1 - k^2)\Omega_{\mathbf{a}},$$

the primitive function of  $(\operatorname{sn}^2 u - 1)^{-2}$  follows.  $\square$

**4.1. Evaluation of integrals.** Write

$$g(s) = \frac{\mathcal{E}_{\mathbf{a}}}{2}s + \frac{\vartheta'}{\vartheta} \left( \frac{s}{\Omega_{\mathbf{a}}}, \tau_0 \right),$$

which is bounded for  $2s + x_0 \in S(\phi, t_\infty, \kappa_0, \delta_0)$  and satisfies  $g((x - x_0)/2) = \mathbf{b}(x)$ . Then, by Lemma 4.2,

$$\begin{aligned} \int_\infty^s \frac{\operatorname{sn} \sigma}{1 - \operatorname{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}} &= \frac{1}{(A_\phi - 1)\Omega_{\mathbf{a}}} \int_\infty^s \left( \frac{\vartheta'}{\vartheta} \left( \frac{\sigma - \alpha_0}{\Omega_{\mathbf{a}}} \right) - \frac{\vartheta'}{\vartheta} \left( \frac{\sigma + \alpha_0}{\Omega_{\mathbf{a}}} \right) \right)_\sigma \frac{d\sigma}{\tilde{\sigma}} \\ &\ll \left| \left( \frac{\vartheta'}{\vartheta} \left( \frac{s - \alpha_0}{\Omega_{\mathbf{a}}} \right) - \frac{\vartheta'}{\vartheta} \left( \frac{s + \alpha_0}{\Omega_{\mathbf{a}}} \right) \right) \frac{1}{\tilde{s}} \right| + \left| \int_\infty^s \left( \frac{\vartheta'}{\vartheta} \left( \frac{\sigma - \alpha_0}{\Omega_{\mathbf{a}}} \right) - \frac{\vartheta'}{\vartheta} \left( \frac{\sigma + \alpha_0}{\Omega_{\mathbf{a}}} \right) \right) \frac{d\sigma}{\tilde{\sigma}^2} \right| \ll s^{-1} \end{aligned}$$

and

$$\begin{aligned} \int_\infty^s \frac{1}{1 - \operatorname{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}} &= \frac{1}{(A_\phi - 1)\Omega_{\mathbf{a}}} \int_\infty^s \left( \mathcal{E}_{\mathbf{a}} \sigma + \frac{\vartheta'}{\vartheta} \left( \frac{\sigma - \alpha_0}{\Omega_{\mathbf{a}}} \right) + \frac{\vartheta'}{\vartheta} \left( \frac{\sigma + \alpha_0}{\Omega_{\mathbf{a}}} \right) \right)_\sigma \frac{d\sigma}{\tilde{\sigma}} \\ &= \frac{1}{(A_\phi - 1)\Omega_{\mathbf{a}}} \int_\infty^s (g(\sigma - \alpha_0) + g(\sigma + \alpha_0))_\sigma \frac{d\sigma}{\tilde{\sigma}} \\ &\ll \left| (g(s - \alpha_0) + g(s + \alpha_0)) \frac{1}{\tilde{s}} \right| + \left| \int_\infty^s (g(\sigma - \alpha_0) + g(\sigma + \alpha_0)) \frac{d\sigma}{\tilde{\sigma}^2} \right| \ll s^{-1} \end{aligned}$$

with  $\tilde{\sigma} = \sigma + x_0/2$ ,  $\tilde{s} = s + x_0/2$  and  $\alpha_0 = (1/4 + \nu_0)\Omega_{\mathbf{a}}$ , which also implies the convergence of these integrals. Then we may write

$$\int_\infty^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} = \int_\infty^x \frac{(4(\theta_0 + \theta_1)\psi_0 - b_0)}{2(A_\phi - \psi_0^2)} \frac{d\xi}{\xi}$$

$$\begin{aligned}
&= 2(\theta_0 + \theta_1) \int_{\infty}^x \frac{\psi_0}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} - \frac{b_0(x_0)}{2} \int_{\infty}^x \frac{1}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} + \frac{4}{\Omega_{\mathbf{a}}} \int_{\infty}^x \frac{\mathbf{b}(\xi)}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} \\
&= \frac{2(\theta_0 + \theta_1)}{A_\phi^{1/2}} \int_{\infty}^s \frac{\operatorname{sn} \sigma}{1 - \operatorname{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}} - \frac{b_0(x_0)}{2A_\phi} \int_{\infty}^s \frac{1}{1 - \operatorname{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}} + \frac{4}{A_\phi \Omega_{\mathbf{a}}} \int_{\infty}^s \frac{g(\sigma)}{1 - \operatorname{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}}, \quad (4.1)
\end{aligned}$$

with  $\sigma = (\xi - x_0)/2$ ,  $s = (x - x_0)/2$ ,  $b_0(x_0) = \beta_0 - 2\mathcal{E}_{\mathbf{a}}\Omega_{\mathbf{a}}^{-1}x_0$ . On the last two lines of (4.1) the first two integrals converge, and consequently, by Proposition 3.2, so the integral containing  $\mathbf{b}(\xi)$  or  $g(\sigma)$ . Let us evaluate it. Set

$$\mathcal{J}_0 := (A_\phi - 1)\Omega_{\mathbf{a}} \int_{\infty}^s \frac{g(\sigma)}{1 - \operatorname{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}} = \int_{\infty}^s (g(\sigma - \alpha_0) + g(\sigma + \alpha_0))_{\sigma} g(\sigma) \frac{d\sigma}{\tilde{\sigma}}.$$

For any sequence  $\{s_\nu\}$  with  $s_\nu = (x_\nu - x_0)/2$ ,

$$\begin{aligned}
&\int_{s_\nu}^s g_\sigma(\sigma + \alpha_0) g(\sigma) \frac{d\sigma}{\tilde{\sigma}} = g(\sigma + \alpha_0) g(\sigma) \tilde{\sigma}^{-1} \Big]_{s_\nu}^s \\
&\quad - \int_{s_\nu}^s g(\sigma + \alpha_0) g_\sigma(\sigma) \frac{d\sigma}{\tilde{\sigma}} + \int_{s_\nu}^s g(\sigma + \alpha_0) g(\sigma) \frac{d\sigma}{\tilde{\sigma}^2} \\
&= - \int_{s_\nu + \alpha_0}^{s + \alpha_0} g(\rho) g_\rho(\rho - \alpha_0) \frac{d\rho}{\tilde{\rho} - \alpha_0} + O(s^{-1}) + O(s_\nu^{-1}) \\
&= - \int_{s_\nu}^s g_\sigma(\sigma - \alpha_0) g(\sigma) \frac{d\sigma}{\tilde{\sigma}} + O(s^{-1}) + O(s_\nu^{-1}),
\end{aligned}$$

which implies  $\mathcal{J}_0 \ll s^{-1}$ . Thus we have the following crucial estimate.

**Proposition 4.3.** *In  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ ,*

$$\begin{aligned}
\int_{\infty}^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} &= 2(\theta_0 + \theta_1) \int_{\infty}^x \frac{\psi_0}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} \\
&\quad - \frac{b_0(x_0)}{2} \int_{\infty}^x \frac{1}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} + \frac{4}{\Omega_{\mathbf{a}}} \int_{\infty}^x \frac{\mathbf{b}(\xi)}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} \ll x^{-1},
\end{aligned}$$

where each integral on the right-hand side is  $O(x^{-1})$ .

Observe that

$$\begin{aligned}
\int_{\infty}^x F_2(\psi_0) \frac{d\xi}{\xi^2} &= \int_{\infty}^x \frac{2(2(\theta_0 - \theta_1)\theta_\infty\psi_0 + (\theta_0 - \theta_1)^2 + \theta_\infty^2)}{(1 - \psi_0^2)(A_\phi - \psi_0^2)} \frac{d\xi}{\xi^2} \\
&= \frac{1}{A_\phi(A_\phi - 1)} \left( 2(\theta_0 - \theta_1)\theta_\infty A_\phi^{1/2} \int_{\infty}^s \left( \frac{A_\phi \operatorname{sn} \sigma}{1 - A_\phi \operatorname{sn}^2 \sigma} - \frac{\operatorname{sn} \sigma}{1 - \operatorname{sn}^2 \sigma} \right) \frac{d\sigma}{\tilde{\sigma}^2} \right. \\
&\quad \left. + ((\theta_0 - \theta_1)^2 + \theta_\infty^2) \int_{\infty}^s \left( \frac{A_\phi}{1 - A_\phi \operatorname{sn}^2 \sigma} - \frac{1}{1 - \operatorname{sn}^2 \sigma} \right) \frac{d\sigma}{\tilde{\sigma}^2} \right).
\end{aligned}$$

In the last line

$$\begin{aligned}
\int_{\infty}^s \frac{A_\phi}{1 - A_\phi \operatorname{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}^2} &= A_\phi \int_{\infty}^s \left( \frac{g(\sigma - \Omega_{\mathbf{a}}/4) + g(\sigma + \Omega_{\mathbf{a}}/4)}{(1 - A_\phi)\Omega_{\mathbf{a}}} + \sigma \right)_{\sigma} \frac{d\sigma}{\tilde{\sigma}^2} \\
&= -A_\phi s^{-1} + O(s^{-2}),
\end{aligned}$$

and the remaining three integrals are  $O(s^{-2})$ . Thus we have the following.

**Proposition 4.4.** In  $\check{S}_{\text{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ ,

$$\int_\infty^x F_2(\psi_0) \frac{d\xi}{\xi^2} = -\frac{2((\theta_0 - \theta_1)^2 + \theta_\infty^2)}{A_\phi - 1} x^{-1} + O(x^{-2}).$$

**4.2. Proof of Proposition 4.1.** By Proposition 3.2, we have

$$h(x) = -\int_\infty^x F_1(\psi_0, b_0) \frac{d\xi}{\xi} + \int_\infty^x \left( F_2(\psi_0) - \frac{3}{2} F_1(\psi_0, b_0)^2 \right) \frac{d\xi}{\xi^2} + \frac{1}{2} \mathcal{J}_1 + O(x^{-2\mu}) \quad (4.2)$$

with

$$\mathcal{J}_1 = \int_\infty^x \frac{1}{A_\phi - \psi_0^2} \int_\infty^\xi (A_\phi - \psi_0^2)(2F_2(\psi_0) - F_1(\psi_0, b_0)^2) \frac{d\xi_1}{\xi_1^2} \frac{d\xi}{\xi}.$$

Then

$$\begin{aligned} \mathcal{J}_1 &= \int_\infty^x \frac{1}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} \cdot \int_\infty^x (A_\phi - \psi_0^2)(2F_2(\psi_0) - F_1(\psi_0, b_0)^2) \frac{d\xi_1}{\xi_1^2} \\ &\quad - \int_\infty^x \int_\infty^\xi \frac{1}{A_\phi - \psi_0^2} \frac{d\xi_1}{\xi_1} \cdot (A_\phi - \psi_0^2)(2F_2(\psi_0) - F_1(\psi_0, b_0)^2) \frac{d\xi}{\xi^2} \ll x^{-2}, \end{aligned}$$

since  $\int_\infty^x (A_\phi - \psi_0^2)^{-1} \xi^{-1} d\xi \ll x^{-1}$ . Insertion of  $\mathcal{J}_1$  into (4.2) combined with Propositions 4.3 and 4.4 yields the desired expression of  $h(x)$ . Thus we have Proposition 4.1.

**4.3. Further calculation of integrals for  $h(x)$ .** In the expression of  $h(x)$  in Proposition 4.1, the second integral becomes

$$\begin{aligned} \int_\infty^x F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} &= \frac{2}{A_\phi} (\theta_0 + \theta_1)^2 \int_\infty^s \left( \frac{1}{(1 - \text{sn}^2 \sigma)^2} - \frac{1}{1 - \text{sn}^2 \sigma} \right) \frac{d\sigma}{\tilde{\sigma}^2} \\ &\quad - \frac{(\theta_0 + \theta_1)}{A_\phi^{3/2}} \int_\infty^s \frac{(b_0(x_0) - 8\Omega_{\mathbf{a}}^{-1} g(\sigma)) \text{sn} \sigma}{(1 - \text{sn}^2 \sigma)^2} \frac{d\sigma}{\tilde{\sigma}^2} \\ &\quad + \frac{1}{8A_\phi^2} \int_\infty^s \frac{b_0(x_0)^2 - 16\Omega_{\mathbf{a}}^{-1} b_0(x_0) g(\sigma) + 64\Omega_{\mathbf{a}}^{-2} g(\sigma)^2}{(1 - \text{sn}^2 \sigma)^2} \frac{d\sigma}{\tilde{\sigma}^2} \\ &= \frac{4(\theta_0 + \theta_1)^2}{3(A_\phi - 1)} x^{-1} + \frac{b_0(x_0)^2}{12A_\phi(A_\phi - 1)} x^{-1} + \frac{4b_0(x_0)}{3A_\phi(A_\phi - 1)\Omega_{\mathbf{a}}} \int_\infty^x \mathfrak{b}(\xi) \frac{d\xi}{\xi^2} \\ &\quad + \frac{16(\theta_0 + \theta_1)}{\Omega_{\mathbf{a}}} \int_\infty^x \frac{\mathfrak{b}(\xi) \psi_0}{(A_\phi - \psi_0^2)^2} \frac{d\xi}{\xi^2} + \frac{16}{\Omega_{\mathbf{a}}^2} \int_\infty^x \frac{\mathfrak{b}(\xi)^2}{(A_\phi - \psi_0^2)^2} \frac{d\xi}{\xi^2} + O(x^{-2}). \quad (4.3) \end{aligned}$$

This is obtained by using

$$\begin{aligned} \int_\infty^s \frac{1}{(1 - \text{sn}^2 \sigma)^2} \frac{d\sigma}{\tilde{\sigma}^2} &= \frac{A_\phi}{3(A_\phi - 1)} s^{-1} + O(s^{-2}), \\ \int_\infty^s \frac{1}{1 - \text{sn}^2 \sigma} \frac{d\sigma}{\tilde{\sigma}^2} &\ll s^{-2}, \quad \int_\infty^s \frac{\text{sn} \sigma}{(1 - \text{sn}^2 \sigma)^2} \frac{d\sigma}{\tilde{\sigma}^2} \ll s^{-2} \end{aligned}$$

and

$$\int_\infty^s \frac{g(\sigma)}{(1 - \text{sn}^2 \sigma)^2} \frac{d\sigma}{\tilde{\sigma}^2} = -\frac{A_\phi}{3(A_\phi - 1)} \int_\infty^s g(\sigma) \frac{d\sigma}{\tilde{\sigma}^2} + O(s^{-2}).$$

In deriving the last equality we note the following:

$$\int_\infty^s (g(\sigma + \alpha_0) + g(\sigma - \alpha_0))_\sigma g(\sigma) \frac{d\sigma}{\tilde{\sigma}^2}, \quad \int_\infty^s (g_\sigma(\sigma + \alpha_0) + g_\sigma(\sigma - \alpha_0))_\sigma g_\sigma(\sigma) \frac{d\sigma}{\tilde{\sigma}^2} \ll s^{-2},$$

which are shown by the same way as in the proof of  $\mathcal{J}_0 \ll s^{-1}$  in Section 4.1. By (4.1) and (4.3),  $h(x)$  is written in the form

$$\begin{aligned} h(x) = & -\frac{2(2\theta_0^2 + 2\theta_1^2 + \theta_\infty^2)}{A_\phi - 1}x^{-1} - 2(\theta_0 + \theta_1) \int_\infty^x \frac{\psi_0}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} \\ & + \frac{b_0(x_0)}{2} \int_\infty^x \frac{1}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} - \frac{4}{\Omega_{\mathbf{a}}} \int_\infty^x \frac{\mathbf{b}(\xi)}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} \\ & - \frac{b_0(x_0)^2}{8A_\phi(A_\phi - 1)}x^{-1} - \frac{2b_0(x_0)}{A_\phi(A_\phi - 1)\Omega_{\mathbf{a}}} \int_\infty^x \frac{\mathbf{b}(\xi)}{\xi^2} d\xi \\ & - \frac{24}{\Omega_{\mathbf{a}}}(\theta_0 + \theta_1) \int_\infty^x \frac{\mathbf{b}(\xi)\psi_0}{(A_\phi - \psi_0^2)^2} \frac{d\xi}{\xi^2} - \frac{24}{\Omega_{\mathbf{a}}^2} \int_\infty^x \frac{\mathbf{b}(\xi)^2}{(A_\phi - \psi_0^2)^2} \frac{d\xi}{\xi^2} + O(x^{-2}). \end{aligned}$$

**4.4. Proof of Theorem 2.3.** Recalling Remark 4.1 and combining

$$\int_\infty^x (A_\phi - \psi_0^2)F_2(\psi_0) \frac{d\xi}{\xi^2} = -2((\theta_0 - \theta_1)^2 + \theta_\infty^2)x^{-1} + O(x^{-2})$$

with the second equality of Proposition 3.2, we obtain Theorem 2.3 by the same argument as in the derivation of Theorems 2.1 and 2.2. Furthermore we have

$$\begin{aligned} b(x) = & b_0(x) + b_0'(x)h(x) - 4(2\theta_0^2 + 2\theta_1^2 + \theta_\infty^2)x^{-1} \\ & - \frac{16(\theta_0 + \theta_1)}{\Omega_{\mathbf{a}}} \int_\infty^x \frac{\mathbf{b}(\xi)\psi_0}{A_\phi - \psi_0^2} \frac{d\xi}{\xi^2} - \frac{16}{\Omega_{\mathbf{a}}^2} \int_\infty^x \frac{\mathbf{b}(\xi)^2}{A_\phi - \psi_0^2} \frac{d\xi}{\xi^2} + O(x^{-2}) \end{aligned}$$

as in Remark 2.3.

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