

# EXPLICIT ERROR TERM OF THE ELLIPTIC ASYMPTOTICS FOR THE FIRST PAINLEVÉ TRANSCENDENTS

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ABSTRACT. For the first Painlevé transcendents Kitaev established an asymptotic representation in terms of the Weierstrass pe-function in cheese-like strips tending along generic directions near the point at infinity. The error term of this asymptotic expression is presented by an explicit formula, which leads to the error bound of exponent  $-1$ .

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*Key words and phrases.* Boutroux ansatz; first Painlevé transcendents; elliptic asymptotics; monodromy data; Weierstrass pe-function.

## 1. INTRODUCTION

The first Painlevé equation

$$y'' = 6y^2 + x \tag{1.1}$$

governs the isomonodromy deformation of the two-dimensional linear system

$$\frac{d\Psi}{d\xi} = \left( (4\xi^4 + x + 2y^2)\sigma_3 - i(4y\xi^2 + x + 2y^2)\sigma_2 - (2y'\xi + \frac{1}{2}\xi^{-1})\sigma_1 \right) \Psi, \tag{1.2}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

([8], [10], [3]). System (1.2) admits the canonical solutions  $\Psi_k(\xi)$  ( $k \in \mathbb{Z}$ ) represented by  $\Psi_k(\xi) = (I + O(\xi^{-1})) \exp((\frac{4}{5}\xi^5 + x\xi)\sigma_3)$  as  $\xi \rightarrow \infty$  through the sector  $-3\pi/10 + \pi k/5 < \arg \xi < \pi/10 + \pi k/5$ , and the Stokes matrices

$$S_{2l+1} = \begin{pmatrix} 1 & s_{2l+1} \\ 0 & 1 \end{pmatrix}, \quad S_{2l} = \begin{pmatrix} 1 & 0 \\ s_{2l} & 1 \end{pmatrix}, \quad l = 0, 1, 2, \dots$$

are defined by  $\Psi_{k+1}(\xi) = \Psi_k(\xi)S_k$ . Each solution of (1.1) is parametrised by  $(s_j)_{1 \leq j \leq 5}$  on the manifold of monodromy data for (1.2), which is a two-dimensional complex manifold in  $\mathbb{C}^5$  given by  $S_1 S_2 S_3 S_4 S_5 = i\sigma_2$  (see [10], [12], [14], [15]). For a general solution thus parametrised, Kapaev [10] obtained asymptotic representations as  $x \rightarrow \infty$  along the Stokes rays  $\arg x = \pi + 2\pi k/5$  ( $k = 0, \pm 1, \pm 2$ ), and subsequent studies on related connection formulas and the Stokes phenomenon are found in [21], [22], [11], [20], [1], and [16], [7] for the  $\tau$ -function. Along generic directions the Boutroux ansatz [2] suggests the asymptotic behaviour of a general solution of (1.1) expressed by the Weierstrass  $\wp$ -function. An approach to such an expression was made by Joshi and Kruskal [9] by the method of multiple-scale expansions, the labelling of the solution by  $(s_j)$  not being considered. Based on the isomonodromy deformation of (1.2), using WKB

analysis, Kitaev [14], [15], and Kapaev and Kitaev [12] presented the elliptic asymptotic representation of a general solution of (1.1) as  $x \rightarrow \infty$  through a cheese-like strip along any direction in the sector  $3\pi/5 < \phi - 2\pi k/5 < \pi$ ,  $k \in \mathbb{Z}$ . For  $k = -2$  the asymptotic result is described as follows [14, p. 593, Section 8], [15, Theorem 2], [12, Theorem 2], in which  $\omega_{\mathbf{a}}$ ,  $\omega_{\mathbf{b}}$  and  $A_\phi$  are, respectively, the periods of an elliptic curve and a unique solution of the Boutroux equations explained in (1) at the end of this section.

**Theorem A.** *Let  $y(x)$  be a solution of (1.1) corresponding to the monodromy data  $(s_j)_{1 \leq j \leq 5}$  such that  $s_1 s_4 \neq 0$ , and let  $\phi$  be such that  $|\phi| < \pi/5$ . Then for any real numbers  $c > 0$  and  $\varepsilon > 0$  there exists a  $\delta > 0$  such that*

$$y(x) = (e^{-i\phi} x)^{1/2} (\wp(e^{i\phi} t - t_0; g_2(\phi), g_3(\phi)) + O(t^{-\delta})), \quad t = \frac{4}{5}(e^{-i\phi} x)^{5/4}, \quad (1.3)$$

as  $x \rightarrow \infty$  through  $\mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$ , where

$$t_0 = \frac{1}{2\pi i} \left( \omega_{\mathbf{a}} \ln(is_1) + \omega_{\mathbf{b}} \ln \frac{s_4}{s_1} \right), \quad (1.4)$$

$\wp(u; g_2, g_3)$  is the  $\wp$ -function such that  $(\wp_u)^2 = 4\wp^3 - g_2\wp - g_3$  with  $g_2(\phi) = -2e^{i\phi}$ ,  $g_3(\phi) = -A_\phi$ , and  $\mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$  the cheese-like strip defined by

$$\left\{ x = e^{i\phi} \left(\frac{5}{4}t\right)^{4/5} \in \mathbb{C} : \operatorname{Re} t > 0, \quad |\operatorname{Im} t| < c, \quad |te^{i\phi} - t_0 + m\omega_{\mathbf{a}} + n\omega_{\mathbf{b}}| > \varepsilon; \quad m, n \in \mathbb{Z} \right\}.$$

*Remark 1.1.* (1) The expression of  $x \in \mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$  (and that in Definition 2.1) is in accordance with [15, Theorem 2, (1.38)], that is, each  $x = e^{i\phi} \left(\frac{5}{4}t\right)^{4/5} \in \mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$  fulfils  $|x| = \left(\frac{5}{4}|t|\right)^{4/5}$  and  $\arg x = \phi + \frac{4}{5} \arg t$ , where  $\arg t \ll |t|^{-1}$  as  $t \rightarrow \infty$ , since  $|\operatorname{Im} t| < c$ . The  $\mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$  contains the centre line  $\arg x = \phi$ .

(2) Let  $\mathcal{S}(c) := \{t \mid \operatorname{Re} t > 0, |\operatorname{Im} t| < c\}$  be the strip corresponding to  $\mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$ . If  $c > 2(|\omega_{\mathbf{a}}| + |\omega_{\mathbf{b}}|)$ , then there exists a string of period parallelograms  $\{L_N\}_{N=1}^\infty$  such that  $\bigcup_{N=1}^\infty L_N \subset \mathcal{S}(c)$  and that, for every  $N$ ,  $(L_N)^{\text{cl}} \cap (L_{N+1})^{\text{cl}}$  is one of  $[v_0^N, v_1^N]$ ,  $[v_1^N, v_2^N]$ ,  $[v_2^N, v_3^N]$ ,  $[v_3^N, v_2^N]$ , where  $v_0^N, v_1^N = v_0^N + \omega_{\mathbf{a}}, v_2^N = v_0^N + \omega_{\mathbf{a}} + \omega_{\mathbf{b}}, v_3^N = v_0^N + \omega_{\mathbf{b}}$  are the vertices of  $L_N$ , and  $(L_N)^{\text{cl}}$  denotes the closure of  $L_N$ .

(3) Practically we may suppose  $\varepsilon$  to be so small that, for all  $(m, n) \in \mathbb{Z}^2$ , the circles  $|te^{i\phi} - t_0 + m\omega_{\mathbf{a}} + n\omega_{\mathbf{b}}| = \varepsilon$  are disjoint.

The leading term of (1.3) depends on the integration constant  $t_0 = t_0((s_j)_{1 \leq j \leq 5})$  only and the other one, which may be called the second integration constant, is hidden in the error term  $O(t^{-\delta})$ . Thus, in treating  $y(x)$  as a general solution, it is preferable to know the explicit error term. This high-order part is also related, say, to the  $\tau$ -function [15, p. 121] or to degeneration to trigonometric asymptotics [15, Section 4]. For the  $\tau$ -function associated with (1.1), by the method of topological recursion, Iwaki [6] obtained a conjectural full-order formal expansion yielding the elliptic expression (see also [7]).

The justification procedure for the asymptotics of  $y(x)$  [15, pp. 105–106, pp. 120–121], [13] is based on the leading terms of (3.1) and of the correction function  $B(\phi, t)$ , the pair of which should depend on both integration constants labelling the solution  $y(x)$ . The

second integration constant is also related to  $B(\phi, t)$  given in the following [15, Theorem 4], where  $J_{\mathbf{a}}$  and  $\vartheta(z, \tau)$  with  $\vartheta'(z, \tau) = (d/dz)\vartheta(z, \tau)$  are as in (1) and (2).

**Theorem B.** *Let  $B(\phi, t) = t((e^{-i\phi}x)^{-3/2}2\mathrm{H}(x, y(x), y'(x)) - A_\phi)$ , where  $\mathrm{H}(x, q, p) = p^2/2 - 2q^3 - xq$  is the Hamiltonian of (1.1). Then as  $t \rightarrow \infty$  the function  $B(\phi, t)$  is bounded uniformly on the set  $\mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$  and is given by*

$$B(\phi, t) = \frac{8}{5\omega_{\mathbf{a}}} \left( \ln \frac{s_1}{s_4} - \frac{5}{4}tJ_{\mathbf{a}} + \pi i + \frac{\vartheta'}{\vartheta} \left( \frac{1}{\omega_{\mathbf{a}}} (te^{i\phi} - t_0) + \nu, \tau \right) \right) + O(t^{-\delta}), \quad (1.5)$$

$\omega_{\mathbf{a}}, t_0, c, \varepsilon$  and  $\delta$  being as in Theorem A.

It is natural to seek explicitly the error term of (1.3). In this paper we present an explicit asymptotic representation of the error term, and this formula leads to the error bound  $O(t^{-1}) = O(x^{-5/4})$  in (1.3). The main results are stated in Theorems 2.1 and 2.2. Corollary 2.3 describes the dependence on the other integration constant  $\beta_0$  in this explicit error term. In Section 4 these results are derived from the  $t^{-\delta}$ -asymptotics of  $y(x)$  in Theorem A and of  $B(\phi, t)$  in Theorem B combined with a system of integral equations equivalent to (1.1), which is constructed in Section 3. Necessary lemmas in our argument are shown in Section 5. The final section is devoted to discussions on our method in deriving the explicit error with the bound  $O(t^{-1})$ , which is quite different from that in [5], [9] and [17].

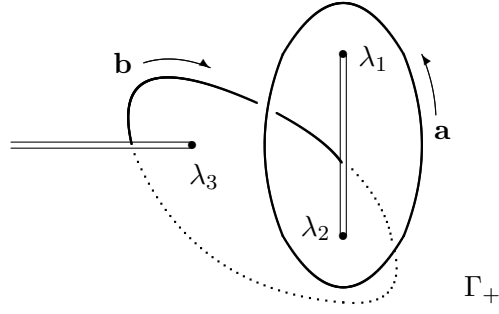
Throughout this paper we use the symbols summed up below (cf. [14], [15]).

(1) For each  $\phi \in \mathbb{R}$ , the condition

$$\operatorname{Re} \int_{\mathbf{c}} w d\lambda = 0,$$

where  $\mathbf{c}$  is any cycle on the elliptic curve  $w^2 = \lambda^3 + \frac{1}{2}e^{i\phi}\lambda + \frac{1}{4}A$ , uniquely determines  $A = A_\phi \in \mathbb{C}$  [15, Lemma 3], [14, Section 7] (The conditions for basic cycles  $\mathbf{c} = \mathbf{c}_1, \mathbf{c}_2$  are the Boutroux equations). Then  $A_\phi$  is continuous in  $\phi$  and smooth for  $\phi \neq \pi + 2\pi k/5$ , ( $k \in \mathbb{Z}$ ), and fulfils the relations  $A_{-\phi} = \overline{A_\phi}$ ,  $A_{\phi+2\pi} = A_\phi$ ,  $A_{\phi-2\pi k/5} = e^{2\pi i k/5} A_\phi$ , and  $A_\pi = -(\frac{2}{3})^{3/2}$ ,  $0.336 < A_0 < 0.380$  [15, Lemma 4], [14, Section 7]. Suppose that  $|\phi| < \pi/5$ , and set  $w = w(A_\phi, \lambda) = \sqrt{\lambda^3 + \frac{1}{2}e^{i\phi}\lambda + \frac{1}{4}A_\phi} = \sqrt{\lambda - \lambda_1} \sqrt{\lambda - \lambda_2} \sqrt{\lambda - \lambda_3}$ . Let  $\mathbf{a}$  and  $\mathbf{b}$  be basic cycles as in Figure 1.1 on the elliptic curve  $\Gamma = \Gamma_+ \cup \Gamma_-$  defined by  $w(A_\phi, \lambda)$  such that the upper and lower sheets  $\Gamma_+$  and  $\Gamma_-$  are glued along the cuts  $[\lambda_1, \lambda_2] \cup [-\infty, \lambda_3]$ ; where the branch points  $\lambda_j = \lambda_j(\phi)$  for  $\phi$  around  $\phi = 0$  are specified in such a way that  $\operatorname{Re} \lambda_1(0) > 0$ ,  $\operatorname{Im} \lambda_1(0) > 0$ ,  $\lambda_2(0) = \overline{\lambda_1(0)}$ ,  $\lambda_3(0) < 0$ , and the branches of  $\sqrt{\lambda - \lambda_j}$  are determined by  $\sqrt{\lambda - \lambda_j} \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$  along the positive real axis on  $\Gamma_+$ . For the cycles  $\mathbf{a}$  and  $\mathbf{b}$  on  $\Gamma$  write

$$\begin{aligned} \omega_{\mathbf{a}} = \omega_{\mathbf{a}}(\phi) &= \frac{1}{2} \int_{\mathbf{a}} \frac{d\lambda}{w}, & \omega_{\mathbf{b}} = \omega_{\mathbf{b}}(\phi) &= \frac{1}{2} \int_{\mathbf{b}} \frac{d\lambda}{w}, \\ J_{\mathbf{a}} = J_{\mathbf{a}}(\phi) &= 2 \int_{\mathbf{a}} w d\lambda, & J_{\mathbf{b}} = J_{\mathbf{b}}(\phi) &= 2 \int_{\mathbf{b}} w d\lambda, \end{aligned}$$

FIGURE 1.1. Cycles **a** and **b**

where  $\omega_{\mathbf{a}}, \omega_{\mathbf{b}}$  fulfil  $\text{Im } \omega_{\mathbf{b}}/\omega_{\mathbf{a}} > 0$ . For  $\phi \neq \pi + 2\pi k/5$  the elliptic curve  $\Gamma$  with the cycles **a**, **b** and the integrals  $\omega_{\mathbf{a}, \mathbf{b}}$  and  $J_{\mathbf{a}, \mathbf{b}}$  may be extended by the use of the substitution  $(\phi, \lambda, w) \mapsto (\phi + 2\pi/5, e^{-4\pi i/5}\lambda, e^{-6\pi i/5}w)$  [15, pp. 95–96].

(2) For  $\tau = \omega_{\mathbf{b}}/\omega_{\mathbf{a}}$  such that  $\text{Im } \tau > 0$ ,

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i z n}$$

is the theta-function [4], [23] with  $\nu = (1 + \tau)/2$ , which satisfies  $\vartheta(z \pm 1, \tau) = \vartheta(z, \tau)$ ,  $\vartheta(z \pm \tau, \tau) = e^{-\pi i(\tau \pm 2z)}\vartheta(z, \tau)$ .

(3) We write  $f \ll g$  or  $g \gg f$  if  $f = O(g)$ , and  $f \asymp g$  if  $f \ll g$  and  $g \ll f$ .

## 2. MAIN RESULTS

To state our results let us define some strips similar to  $\mathcal{D}_{(s_j)}(\phi, c, \varepsilon)$ . Recall the phase shift  $t_0$  given by (1.4), and the zeros  $\lambda_j$  ( $1 \leq j \leq 3$ ) of  $w(A_\phi, \lambda)^2$  specified as in (1).

**Definition 2.1.** For given  $t_\infty > 0$ ,  $c > 0$  and small  $\varepsilon > 0$ , set

$$\begin{aligned} \mathcal{D}(\phi, t_\infty, c, \varepsilon) &= \{x = e^{i\phi}(\frac{5}{4}t)^{4/5} \in \mathbb{C} : \\ &\quad \text{Re } t > t_\infty, |\text{Im } t| < c, |e^{i\phi}t - t_0 - m\omega_{\mathbf{a}} - n\omega_{\mathbf{b}}| > \varepsilon, m, n \in \mathbb{Z}\}, \end{aligned}$$

$$\check{\mathcal{D}}(\phi, t_\infty, c, \varepsilon) = \mathcal{D}(\phi, t_\infty, c, \varepsilon) \setminus \bigcup_{1 \leq j \leq 3; m, n \in \mathbb{Z}} \mathcal{V}_{j, m, n}(\varepsilon),$$

$$\mathcal{V}_{j, m, n}(\varepsilon) = \{x = e^{i\phi}(\frac{5}{4}t)^{4/5} \in \mathbb{C} : |e^{i\phi}(t - t_j) - t_0 - m\omega_{\mathbf{a}} - n\omega_{\mathbf{b}}| \leq \varepsilon\},$$

where  $t_j$  ( $1 \leq j \leq 3$ ) are such that  $\wp(e^{i\phi}t_j - t_0; g_2(\phi), g_3(\phi)) = \lambda_j$  (for  $x = e^{i\phi}(\frac{5}{4}t)^{4/5}$  in these domains, cf. Remark 1.1).

Let  $t_j^{m, n} \in \mathcal{S}(t_\infty, c) := \{t \mid \text{Re } t > t_\infty, |\text{Im } t| < c\}$  ( $j = 1, 2, 3, \infty$ ) be such that  $t_j^{m, n} = t_j + e^{-i\phi}(t_0 + m\omega_{\mathbf{a}} + n\omega_{\mathbf{b}})$  if  $1 \leq j \leq 3$ , and that  $t_\infty^{m, n} = e^{-i\phi}(t_0 + m\omega_{\mathbf{a}} + n\omega_{\mathbf{b}})$  if  $j = \infty$ . We may suppose  $\varepsilon$  so small that all the circles  $|t - t_j^{m, n}| = \varepsilon$  ( $j = 1, 2, 3, \infty; m, n \in \mathbb{Z}$ ) are disjoint. Let  $l(t_j^{m, n}) \in \mathcal{S}(t_\infty, c)$  ( $j = 1, 2, 3, \infty$ ) be the lines defined by  $t = \text{Re } t_j^{m, n} + i\eta$ ,  $\eta \geq \text{Im } t_j^{m, n}$  if  $\text{Im } t_j^{m, n} \geq 0$  (respectively,  $\eta \leq \text{Im } t_j^{m, n}$  if  $\text{Im } t_j^{m, n} < 0$ ), and let  $l^x(t_j^{m, n})$  be the images of  $l(t_j^{m, n})$  under the mapping  $x = x(t) = e^{i\phi}(\frac{5}{4}t)^{4/5}$ .

**Definition 2.2.** For  $t_\infty, c, \varepsilon$  as in Definition 2.1,  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$  denotes  $\check{\mathcal{D}}(\phi, t_\infty, c, \varepsilon)$  having cuts along  $l^x(t_j^{m,n})$  for all  $j = 1, 2, 3, \infty$  and  $m, n \in \mathbb{Z}$ , where some local segments contained in  $l^x(t_j^{m,n})$  are replaced with suitable arcs, if necessary, not to touch the image of another small circle  $|t - t_{j',m',n'}| = \varepsilon$  with  $(j', m', n') \neq (j, m, n)$  under the mapping  $x(t)$ .

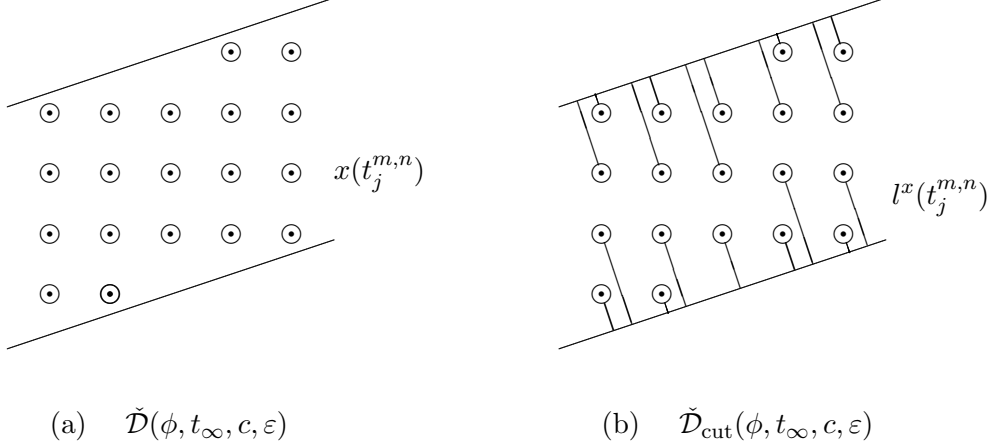


FIGURE 2.1.  $\check{\mathcal{D}}(\phi, t_\infty, c, \varepsilon)$  and  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$

These domains are strips lying in the  $x$ -plane. For a function  $f(x)$  we may deal with  $\hat{f}(t) = f(x(t))$  with  $t = \frac{4}{5}(e^{-i\phi}x)^{5/4}$  or  $\hat{f}(z) = f(x(z))$  with  $z = e^{i\phi}t = \frac{4}{5}e^{i\phi}(e^{-i\phi}x)^{5/4}$  as  $x \rightarrow \infty$ , say, through  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ , and then we write, in short,  $f(t)$  or  $f(z)$  in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ .

Suppose that  $|\phi| < \pi/5$ . Let  $y(x)$  be a solution of (1.1) corresponding to  $(s_j)_{1 \leq j \leq 5}$  such that  $s_1 s_4 \neq 0$ . Then we have the following.

**Theorem 2.1.** *Let  $c > 0$  and  $\varepsilon > 0$  be given positive numbers as in Theorem A. Then*

$$y(x) = (e^{-i\phi}x)^{1/2} (\wp(e^{i\phi}t - t_0; g_2(\phi), g_3(\phi)) + O(t^{-1}))$$

as  $x = e^{i\phi}(\frac{5}{4}t)^{4/5} \rightarrow \infty$  through  $\mathcal{D}(\phi, t_\infty, c, \varepsilon)$ , where  $t_\infty$  is sufficiently large.

To deal with the error term explicitly, let us write (1.3) in the form

$$y(x) = (e^{-i\phi}x)^{1/2} \wp(e^{i\phi}t - t_0 + h(e^{i\phi}t); g_2(\phi), g_3(\phi)).$$

Set  $z = e^{i\phi}t$  and write

$$\begin{aligned} \mathbf{p} &= \mathbf{p}(z) = \wp(z - t_0; g_2(\phi), g_3(\phi)), \\ \beta &= \beta(z) = \frac{8e^{i\phi}}{5\omega_{\mathbf{a}}} \left( \beta_0 - \frac{5}{4}e^{-i\phi} J_{\mathbf{a}}(z - t_0) + \frac{\wp'}{\wp} \left( \frac{z - t_0}{\omega_{\mathbf{a}}} + \nu, \tau \right) \right), \\ \beta_0 &= \ln \frac{s_1}{s_4} - \frac{5}{4}e^{-i\phi} J_{\mathbf{a}} t_0 + \pi i. \end{aligned}$$

The quantity  $\beta(z) = e^{i\phi} B_{\text{as}}(\phi, t)$  is the leading term of the correction function  $e^{i\phi} B(\phi, t)$  in Theorem B, and  $\beta(z)$  and  $\mathbf{p}(z)$  are bounded uniformly in  $\mathcal{D}(\phi, t_\infty, c, \varepsilon)$ . Then we have the following explicit expression of  $h(z)$ .

**Theorem 2.2.** *Let  $P(\lambda) = 4w(A_\phi, \lambda)^2 = 4\lambda^3 + 2e^{i\phi}\lambda + A_\phi$ , and let  $c, \varepsilon, \delta$  be as in Theorem A. Then as  $x = e^{i\phi}(\frac{5}{4}e^{-i\phi}z)^{4/5} \rightarrow \infty$  through  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ ,*

$$h(z) = \int_\infty^z F(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} - \int_\infty^z (F(\mathbf{p}, \beta)^2 + G(\mathbf{p}, \beta)) \frac{d\zeta}{\zeta^2} - \frac{1}{10} \int_\infty^z K(\mathbf{p}, \beta) \mathcal{I}(\zeta) \frac{d\zeta}{\zeta} + O(z^{-1-\delta}),$$

in which

$$F(\mathbf{p}, \beta) = \frac{\beta}{2P(\mathbf{p})} - \frac{2\mathbf{p}}{5\sqrt{P(\mathbf{p})}}, \quad G(\mathbf{p}, \beta) = \frac{\beta^2}{8P(\mathbf{p})^2},$$

$$K(\mathbf{p}, \beta) = 2(4e^{i\phi}\mathbf{p} + 3A_\phi)F(\mathbf{p}, \beta) - \beta, \quad \mathcal{I}(z) = \int_\infty^z \frac{1}{P(\mathbf{p})} \frac{d\zeta}{\zeta}$$

with  $\mathbf{p} = \mathbf{p}(\zeta)$ ,  $\beta = \beta(\zeta)$ , and in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$  the integrals converge and are evaluated as

$$\mathcal{I}(z) \ll z^{-1}, \quad \int_\infty^z F(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} \ll z^{-1},$$

$$\int_\infty^z (F(\mathbf{p}, \beta)^2 + G(\mathbf{p}, \beta)) \frac{d\zeta}{\zeta^2} \ll z^{-1}, \quad \int_\infty^z K(\mathbf{p}, \beta) \mathcal{I}(\zeta) \frac{d\zeta}{\zeta} \ll z^{-1}.$$

*Remark 2.1.* For the sake of convenience the integrals are considered in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ , which is simply connected, to avoid the possible multi-valuedness around poles of the integrands. Then  $h(z)$  is considered substantially along the ray  $\arg x = \phi$  and expressed by the  $\wp$ -function with  $(g_2(\phi), g_3(\phi)) = (-2e^{i\phi}, -A_\phi)$ .

The integration constant  $\beta_0$  appears in  $h(z)$  as described in the following.

**Corollary 2.3.** *In  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$  we have  $zh(z) = h_0\beta_0^2 + h_1(z)\beta_0 + h_2(z) + O(z^{-\delta})$  with  $h_1(z) = O(1)$ ,  $h_2(z) = O(1)$  and*

$$h_0 = -\frac{24}{5\omega_{\mathbf{a}}^2} e^{2i\phi} (8e^{3i\phi} + 27A_\phi^2)^{-1}.$$

*Remark 2.2.* Along any direction in the sector  $3\pi/5 < \phi - 2\pi k/5 < \pi$ ,  $k \in \mathbb{Z}$ , the formulas of the main results are written in terms of  $t_0$  and  $\beta_0$  with  $(s_{2-2k}, s_{5-2k}/s_{2-2k})$  in place of  $(s_1, s_4/s_1)$  (cf. [15, Theorems 2 and 4]). This discontinuity of integration constants may also be considered as a nonlinear Stokes phenomenon [3, p. 379].

*Remark 2.3.* For a given simply connected unbounded subdomain  $\mathcal{D}_0 \subset \check{\mathcal{D}}(\phi, t_\infty, c, \varepsilon)$ , Theorem 2.2 and Corollary 2.3 are also valid as  $x \rightarrow \infty$  through  $\mathcal{D}_0$ .

## 3. SYSTEM OF EQUATIONS AND INTEGRAL REPRESENTATIONS

Let  $c$ ,  $\varepsilon$  and  $\delta$  be as in Theorem A. The number  $t_\infty$  is retaken if necessary, being denoted by the same letter  $t_\infty$  in each appearance in our argument. In the proofs of our theorems we may suppose that  $0 < \delta < 1$ .

By the change of variables  $t = \frac{4}{5}(e^{-i\phi}x)^{5/4}$ ,  $y = (e^{-i\phi}x)^{1/2}v$ , equation (1.1) or  $(d/dx)((y')^2 - 4y^3 - 2xy) = -2y$  is taken to

$$t \frac{d}{dt} a_\phi + \frac{6}{5} a_\phi = -\frac{8}{5} e^{i\phi} v$$

with

$$a_\phi = e^{-2i\phi} (v_t + (2/5)t^{-1}v)^2 - 4v^3 - 2e^{i\phi}v,$$

which is the Lagrangian or the Hamiltonian  $2(e^{-i\phi}x)^{-3/2}H(x, y(x), y'(x))$  appearing in Theorem B, [15, Theorem 4]. Setting  $a_\phi = A_\phi + t^{-1}B(\phi, t)$  with  $e^{i\phi}t = z$ ,  $e^{i\phi}B(\phi, t) = b$ , we have the system of equations

$$\begin{aligned} (v_z + (2/5)z^{-1}v)^2 &= 4v^3 + 2e^{i\phi}v + A_\phi + z^{-1}b, \\ b_z &= -\frac{8}{5}e^{i\phi}v - \frac{6}{5}A_\phi - \frac{1}{5}z^{-1}b. \end{aligned} \quad (3.1)$$

Note that (1.5) is written as  $B(\phi, t) = B_{\text{as}}(\phi, t) + O(t^{-\delta})$ . The system

$$\begin{aligned} \mathbf{p}_z^2 &= 4\mathbf{p}^3 + 2e^{i\phi}\mathbf{p} + A_\phi, \\ \beta_z &= -\frac{8}{5}e^{i\phi}\mathbf{p} - \frac{6}{5}A_\phi \end{aligned} \quad (3.2)$$

admits a solution  $(\mathbf{p}, \beta) = (\mathbf{p}(z), \beta(z)) = (\wp(z - t_0; g_2(\phi), g_3(\phi)), e^{i\phi}B_{\text{as}}(\phi, e^{-i\phi}z))$ , where validity of the second equation is due to [15, Proposition 6, (3.13)]. System (3.2) is, at least formally, an approximation to (3.1).

**Proposition 3.1.** *For  $y(x)$  given by (1.3) set  $\mathbf{p}(z + h(z)) = (e^{-i\phi}x)^{-1/2}y(x)$  with  $e^{i\phi}t = z$  in  $\check{D}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ . Then  $(v, b) = (\mathbf{p}(z + h(z)), e^{i\phi}B(\phi, e^{-i\phi}z))$  solves (3.1).*

By (1.3) and [15, Theorem 4] with  $b(z) - \beta(z) = e^{i\phi}(B(\phi, e^{-i\phi}z) - B_{\text{as}}(\phi, e^{-i\phi}z))$ , we have the following.

**Proposition 3.2.** *In  $\check{D}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ ,  $h(z) \ll z^{-\delta}$ , and  $b(z)$  is bounded and satisfies  $b(z) - \beta(z) \ll z^{-\delta}$ .*

Recall that  $P(\lambda) = 4\lambda^3 + 2e^{i\phi}\lambda + A_\phi$ . Let us insert  $v = \mathbf{p}_{(h)}(z) := \mathbf{p}(z + h(z))$  into the first equation of (3.1). Observing that  $v_z = (1 + h')\mathbf{p}'(z + h) = (1 + h')\sqrt{P(\mathbf{p}_{(h)})}$ , and that  $(1 + h')\sqrt{P(\mathbf{p}_{(h)})} + (2/5)z^{-1}\mathbf{p}_{(h)} = \sqrt{P(\mathbf{p}_{(h)})}(1 + z^{-1}bP(\mathbf{p}_{(h)})^{-1})^{1/2}$ , we have, in  $\check{D}(\phi, t_\infty, c, \varepsilon)$ ,

$$h' = \left( \frac{b}{2P(\mathbf{p}_{(h)})} - \frac{2\mathbf{p}_{(h)}}{5\sqrt{P(\mathbf{p}_{(h)})}} \right) z^{-1} - \frac{b^2}{8P(\mathbf{p}_{(h)})^2} z^{-2} + O(z^{-3}).$$

Furthermore, by  $\mathbf{p}_{(h)}(z) = \mathbf{p}(z) + h\mathbf{p}'(z) + O(h^2)$ ,

$$h' = F(\mathbf{p}, b)z^{-1} - G(\mathbf{p}, b)z^{-2} + F_v(\mathbf{p}, b)\mathbf{p}'hz^{-1} + O(z^{-1}(|z^{-1}| + |h|)^2) \quad (3.3)$$

with  $F(v, b) = \frac{1}{2}bP(v)^{-1} - \frac{2}{5}vP(v)^{-1/2}$ ,  $G(v, b) = \frac{1}{8}b^2P(v)^{-2}$  as in Theorem 2.2. Write  $\chi := b - \beta$ . Then the second equations of (3.1) and (3.2) yield

$$\begin{aligned}\chi' &= -\frac{8}{5}e^{i\phi}(\mathbf{p}_{(h)}(z) - \mathbf{p}(z)) - \frac{b(z)}{5}z^{-1} \\ &= -\frac{8}{5}e^{i\phi}\left(\mathbf{p}'h + \frac{\mathbf{p}''}{2!}h^2 + \cdots + \frac{\mathbf{p}^{(m)}}{m!}h^m + E_m h^{m+1}\right) - \frac{b}{5}z^{-1}\end{aligned}$$

for any positive integer  $m$ , where  $E_m \ll 1$  in  $\check{D}(\phi, t_\infty, c, \varepsilon)$ . Let  $\{z_n\} \subset \check{D}(\phi, t_\infty, c, \varepsilon)$  be any sequence such that  $z_n \rightarrow \infty$ . Then integration by parts leads to

$$\begin{aligned}\chi(z) - \chi(z_n) &= -\frac{8}{5}e^{i\phi}\left[\mathbf{p}h + \frac{\mathbf{p}'}{2}h^2 + \cdots + \frac{\mathbf{p}^{(m-1)}}{m!}h^m\right]_{z_n}^z \\ &\quad + \frac{8}{5}e^{i\phi}\int_{z_n}^z\left(\left(\mathbf{p} + \mathbf{p}'h + \cdots + \frac{\mathbf{p}^{(m-1)}}{(m-1)!}h^{m-1}\right)h' - E_m h^{m+1}\right)d\zeta - \frac{1}{5}\int_{z_n}^z(\beta + \chi)\frac{d\zeta}{\zeta}.\end{aligned}$$

By (3.3),  $h' = (F(\mathbf{p}, \beta) + \frac{1}{2}\chi P(\mathbf{p})^{-1} + \tilde{R})z^{-1}$  with  $\tilde{R} \ll |z^{-1}| + |h|$ , and hence the sum of the integrals on the right-hand side is

$$\frac{1}{5}\int_{z_n}^z(8e^{i\phi}F(\mathbf{p}, \beta)\mathbf{p} - \beta)\frac{d\zeta}{\zeta} + \frac{1}{5}\int_{z_n}^z R(h, \chi)\frac{d\zeta}{\zeta} - \frac{8}{5}e^{i\phi}\int_{z_n}^z E_m h^{m+1}d\zeta,$$

with

$$\begin{aligned}R(h, \chi) &:= 8e^{i\phi}\left(\mathbf{p}' + \cdots + \frac{\mathbf{p}^{(m-1)}}{(m-1)!}h^{m-2}\right)\left(F(\mathbf{p}, \beta) + \frac{\chi}{2P(\mathbf{p})} + \tilde{R}\right)h \\ &\quad + 8e^{i\phi}\mathbf{p}\left(\frac{\chi}{2P(\mathbf{p})} + \tilde{R}\right) - \chi \ll |h| + |\chi| + |\zeta^{-1}|.\end{aligned}\tag{3.4}$$

Now suppose that, for a positive number  $\mu$  satisfying  $\delta \leq \mu \leq 1$ ,

$$h(z) \ll z^{-\mu}\tag{3.5}$$

in  $\check{D}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ . By Proposition 3.2 this supposition is true for  $\mu = \delta$ , and  $\chi(z) \ll z^{-\delta}$ . Choose  $m$  such that  $\delta(m+1) \geq 3$ . The passage to the limit  $z_n \rightarrow \infty$  leads to

$$\chi + \frac{8}{5}e^{i\phi}\mathbf{p}h = \frac{1}{5}\int_{\infty}^z(8e^{i\phi}F(\mathbf{p}, \beta)\mathbf{p} - \beta)\frac{d\zeta}{\zeta} + \frac{1}{5}\int_{\infty}^z R(h, \chi)\frac{d\zeta}{\zeta} + O(z^{-2\mu}),\tag{3.6}$$

in which the convergence of

$$\int_{\infty}^z(8e^{i\phi}F(\mathbf{p}, \beta)\mathbf{p} - \beta)\frac{d\zeta}{\zeta} \ll z^{-\delta}$$

is guaranteed by the absolute convergence of  $\int_{\infty}^z R(h, \chi)\zeta^{-1}d\zeta \ll z^{-\delta}$  (cf. (3.4) and Proposition 3.2). Under (3.5), observing that, in (3.3),

$$-G(\mathbf{p}, b)z^{-2} + F_v(\mathbf{p}, b)\mathbf{p}'hz^{-1} = -G(\mathbf{p}, \beta)z^{-2} + F_v(\mathbf{p}, \beta)\mathbf{p}'hz^{-1} + O(z^{-1-\mu-\delta}),$$

and that

$$\begin{aligned}\int_{z_n}^z F_v(\mathbf{p}, \beta)\mathbf{p}'h\frac{d\zeta}{\zeta} &= \int_{z_n}^z\left(F(\mathbf{p}, \beta)' - \frac{\beta'}{2P(\mathbf{p})}\right)h\frac{d\zeta}{\zeta} \\ &= \left[F(\mathbf{p}, \beta)h\zeta^{-1}\right]_{z_n}^z - \int_{z_n}^z\left(F(\mathbf{p}, \beta)h' + \frac{\beta'h}{2P(\mathbf{p})}\right)\frac{d\zeta}{\zeta} + \left[O(z^{-1-\mu})\right]_{z_n}^z\end{aligned}$$

$$= - \int_{z_n}^z \left( F(\mathbf{p}, \beta)^2 \zeta^{-1} + \frac{\beta' h}{2P(\mathbf{p})} \right) \frac{d\zeta}{\zeta} + \left[ O(z^{-1-\delta}) \right]_{z_n}^z,$$

we may apply a similar argument to (3.3) with  $F(\mathbf{p}, b) = F(\mathbf{p}, \beta) + \frac{1}{2}\chi P(\mathbf{p})^{-1}$ , and the convergence of  $\int_{\infty}^z F(\mathbf{p}, \beta) \zeta^{-1} d\zeta$  follows. Thus we have the following relations, in which the second equation follows from (3.6) and (3.2).

**Proposition 3.3.** *Under the supposition (3.5), in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ ,*

$$\begin{aligned} h &= \int_{\infty}^z F(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} - \int_{\infty}^z (F(\mathbf{p}, \beta)^2 + G(\mathbf{p}, \beta)) \frac{d\zeta}{\zeta^2} + \int_{\infty}^z \frac{\chi - \beta' h}{2P(\mathbf{p})} \frac{d\zeta}{\zeta} + O(z^{-\mu-\delta}), \\ \chi - \beta' h &= \frac{6}{5} A_{\phi} h + \frac{1}{5} \int_{\infty}^z H(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} + \frac{1}{5} \int_{\infty}^z R(h, \chi) \frac{d\zeta}{\zeta} + O(z^{-2\mu}), \end{aligned}$$

in which  $H(\mathbf{p}, \beta) = 8e^{i\phi} F(\mathbf{p}, \beta) \mathbf{p} - \beta$ , every integral converges, and

$$\begin{aligned} \int_{\infty}^z F(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} &\ll z^{-\delta}, & \int_{\infty}^z (F(\mathbf{p}, \beta)^2 + G(\mathbf{p}, \beta)) \frac{d\zeta}{\zeta^2} &\ll z^{-1}, & \int_{\infty}^z \frac{\chi - \beta' h}{2P(\mathbf{p})} \frac{d\zeta}{\zeta} &\ll z^{-\delta}, \\ \int_{\infty}^z H(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} &\ll z^{-\delta}, & \int_{\infty}^z R(h, \chi) \frac{d\zeta}{\zeta} &\ll z^{-\delta}. \end{aligned}$$

#### 4. PROOFS OF THE MAIN RESULTS

By Proposition 3.3,

$$h - \frac{3}{5} A_{\phi} \int_{\infty}^z \frac{h}{P(\mathbf{p})} \frac{d\zeta}{\zeta} = \int_{\infty}^z F(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} - \int_{\infty}^z (F(\mathbf{p}, \beta)^2 + G(\mathbf{p}, \beta)) \frac{d\zeta}{\zeta^2} + I_1 + O(z^{-\mu-\delta})$$

with

$$I_1 = \frac{1}{10} \int_{\infty}^z \frac{1}{P(\mathbf{p})} \int_{\infty}^{\zeta} (H(\mathbf{p}, \beta) + R(h, \chi)) \frac{d\zeta_1}{\zeta_1} \frac{d\zeta}{\zeta}$$

in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$  since  $\mu \geq \delta$ . Note that  $\int_{\infty}^z P(\mathbf{p})^{-1} \zeta^{-1} d\zeta \ll z^{-1}$  by Lemma 5.5 shown later, and  $\int_{\infty}^z H(\mathbf{p}, \beta) \zeta^{-1} d\zeta \ll z^{-\delta}$ ,  $\int_{\infty}^z R(h, \chi) \zeta^{-1} d\zeta \ll z^{-\delta}$  by Proposition 3.3 and (3.4). Then integration by parts leads to

$$\begin{aligned} 10I_1 &= \int_{\infty}^z \frac{1}{P(\mathbf{p})} \frac{d\zeta}{\zeta} \int_{\infty}^z (H(\mathbf{p}, \beta) + R(h, \chi)) \frac{d\zeta}{\zeta} \\ &\quad - \int_{\infty}^z \int_{\infty}^{\zeta} \frac{1}{P(\mathbf{p})} \frac{d\zeta_1}{\zeta_1} (H(\mathbf{p}, \beta) + R(h, \chi)) \frac{d\zeta}{\zeta} \\ &= - \int_{\infty}^z H(\mathbf{p}, \beta) \int_{\infty}^{\zeta} \frac{1}{P(\mathbf{p})} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta}{\zeta} + O(z^{-1-\delta}). \end{aligned} \tag{4.1}$$

Furthermore, by (3.3), (3.5), Lemma 5.5 and Proposition 3.2,

$$\begin{aligned} \int_{\infty}^z \frac{h}{P(\mathbf{p})} \frac{d\zeta}{\zeta} &= h \int_{\infty}^z \frac{1}{P(\mathbf{p})} \frac{d\zeta}{\zeta} - \int_{\infty}^z h' \int_{\infty}^{\zeta} \frac{1}{P(\mathbf{p})} \frac{d\zeta_1}{\zeta_1} d\zeta \\ &= - \int_{\infty}^z \left( F(\mathbf{p}, \beta) + \frac{\chi}{2P(\mathbf{p})} \right) \int_{\infty}^{\zeta} \frac{1}{P(\mathbf{p})} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta}{\zeta} + O(z^{-1-\mu}) \\ &= - \int_{\infty}^z F(\mathbf{p}, \beta) \int_{\infty}^{\zeta} \frac{1}{P(\mathbf{p})} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta}{\zeta} + O(z^{-1-\delta}). \end{aligned} \tag{4.2}$$

Using (4.1) and (4.2) we obtain, under supposition (3.5),

$$\begin{aligned} h &= \int_{\infty}^z F(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} - \int_{\infty}^z (F(\mathbf{p}, \beta)^2 + G(\mathbf{p}, \beta)) \frac{d\zeta}{\zeta^2} \\ &\quad - \frac{1}{10} \int_{\infty}^z K(\mathbf{p}, \beta) \mathcal{I}(\zeta) \frac{d\zeta}{\zeta} + O(z^{-\mu-\delta}) \end{aligned} \quad (4.3)$$

in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ , in which the implied constant possibly depends on  $\mu$ , and  $K(\mathbf{p}, \beta) = H(\mathbf{p}, \beta) + 6A_{\phi}F(\mathbf{p}, \beta) = 2(4e^{i\phi}\mathbf{p} + 3A_{\phi})F(\mathbf{p}, \beta) - \beta$ ,  $\mathcal{I}(z) = \int_{\infty}^z P(\mathbf{p})^{-1}\zeta^{-1}d\zeta$ . The integrals on the right-hand side of (4.3) satisfy

$$\begin{aligned} \int_{\infty}^z F(\mathbf{p}, \beta) \frac{d\zeta}{\zeta} &\ll z^{-1}, \quad \int_{\infty}^z (F(\mathbf{p}, \beta)^2 + G(\mathbf{p}, \beta)) \frac{d\zeta}{\zeta^2} \ll z^{-1}, \\ \int_{\infty}^z K(\mathbf{p}, \beta) \mathcal{I}(\zeta) \frac{d\zeta}{\zeta} &\ll z^{-1}, \quad \mathcal{I}(z) \ll z^{-1}, \end{aligned} \quad (4.4)$$

in which the first nontrivial estimate follows from Lemmas 5.5 and 5.4 with  $\mathbf{p}/\sqrt{P(\mathbf{p})} = \mathbf{p}\mathbf{p}'/P(\mathbf{p})$  and  $\beta(z) = \frac{8}{5}e^{i\phi}\omega_{\mathbf{a}}^{-1}(\beta_0 + \mathbf{b}(z - t_0))$  (cf. (5.4)).

**4.1. Derivation of Theorem 2.2 and Corollary 2.3.** To prove Theorem 2.2 let us start with (1.3) and (1.5) of Theorems A and B with given  $c > 0$ ,  $\varepsilon > 0$  and a small  $\delta > 0$ . By (4.4) the sum of the integrals on the right-hand side of (4.3) is  $O(z^{-1})$  in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$  for sufficiently large  $t_{\infty}$ . Note that (3.5) is valid for  $\mu = \delta$ , i.e.  $h(z) \ll z^{-\delta}$ . Then, by (4.3) with  $\mu = \delta$ , we have  $h(z) \ll |z^{-1}| + |z^{-2\delta}|$ . If  $2\delta < 1$ , then (3.5) is valid for  $\mu = 2\delta$ , which implies asymptotic formula (4.3) with the error bound  $O(z^{-3\delta})$  and  $h(z) \ll |z^{-1}| + |z^{-3\delta}|$ . For  $m_0$  such that  $m_0\delta < 1 \leq (m_0 + 1)\delta$ ,  $m_0$ -times repetition of this procedure results in (4.3) with  $\mu = m_0\delta$ , which yields (3.5) with  $\mu = 1$ . Then (4.3) with  $\mu = 1$  is valid in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ , implying Theorem 2.2. Calculation of the coefficient of  $\beta_0^2$  by the use of Lemma 5.5 leads us to Corollary 2.3.

**4.2. Derivation of Theorem 2.1.** Theorem 2.2 implies  $h(z) \ll z^{-1}$  in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ . Note that

$$\begin{aligned} &(e^{-i\phi}x)^{-1/2}y(x) - \wp(e^{i\phi}t - t_0; g_2(\phi), g_3(\phi)) \\ &= \wp(e^{i\phi}t - t_0 + h(e^{i\phi}t); g_2(\phi), g_3(\phi)) - \wp(e^{i\phi}t - t_0; g_2(\phi), g_3(\phi)) \\ &\ll \wp'(e^{i\phi}t - t_0; g_2(\phi), g_3(\phi))h(e^{i\phi}t) \ll t^{-1} \end{aligned}$$

in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ , and that the function on the first line is holomorphic in  $\mathcal{D}(\phi, t_{\infty}, c, \varepsilon)$ . By using the maximal modulus principle in excluded discs around  $t_j^{m,n}$  ( $1 \leq j \leq 3$ ;  $m, n \in \mathbb{Z}$ ), we obtain Theorem 2.1.

## 5. LEMMAS ON PRIMITIVE FUNCTIONS

It remains to show the lemmas on primitive functions used in the proofs of the main results. Recall that  $P(\lambda) = 4w(A_{\phi}, \lambda)^2$  as in Theorem 2.2, and let us write

$$P(\lambda) = 4\lambda^3 - g_2\lambda - g_3 = 4(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3), \quad g_2 = -2e^{i\phi}, \quad g_3 = -A_{\phi},$$

$$\lambda_1 = \wp(\omega_1), \quad \lambda_2 = \wp(\omega_2), \quad \lambda_3 = \wp(\omega_3),$$

where

$$\omega_1 = \frac{1}{2}\omega_{\mathbf{a}}, \quad \omega_3 = \frac{1}{2}\omega_{\mathbf{b}}, \quad \omega_1 + \omega_2 + \omega_3 = 0.$$

Then

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 0, & \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 &= -g_2/4, & \lambda_1\lambda_2\lambda_3 &= g_3/4, \\ (\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2(\lambda_1 - \lambda_2)^2 &= \frac{1}{16}(g_2^3 - 27g_3^2), \end{aligned}$$

and  $\omega_{\mathbf{a}}$  and  $J_{\mathbf{a}}$  are also written in the form

$$\omega_{\mathbf{a}} = \int_{\mathbf{a}} \frac{d\lambda}{\sqrt{P(\lambda)}}, \quad J_{\mathbf{a}} = \int_{\mathbf{a}} \sqrt{P(\lambda)} d\lambda.$$

**Lemma 5.1.** *Let*

$$\begin{aligned} \gamma_1 &= (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3), & \gamma_2 &= (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1), & \gamma_3 &= (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2), \\ \gamma_0 &= 5(g_2^3 - 27g_3^2)^{-1} \end{aligned}$$

and  $\wp(z) = \wp(z; g_2, g_3)$ . Then

$$\begin{aligned} \int_0^z \frac{dz}{P(\wp(z))} &= -\frac{1}{4\omega_{\mathbf{a}}} \left( \gamma_1^{-2} \left( \frac{5J_{\mathbf{a}}}{2g_2} z + \frac{\wp'}{\wp} \left( \frac{z}{\omega_{\mathbf{a}}} + \frac{\tau}{2}, \tau \right) \right) + \gamma_2^{-2} \left( \frac{5J_{\mathbf{a}}}{2g_2} z + \frac{\wp'}{\wp} \left( \frac{z}{\omega_{\mathbf{a}}}, \tau \right) \right) \right. \\ &\quad \left. + \gamma_3^{-2} \left( \frac{5J_{\mathbf{a}}}{2g_2} z + \frac{\wp'}{\wp} \left( \frac{z}{\omega_{\mathbf{a}}} + \frac{1}{2}, \tau \right) \right) \right) + C_0, \\ \int_0^z \frac{dz}{P(\wp(z))^2} &= \gamma_0 z - \frac{1}{96\omega_{\mathbf{a}}} \left( \gamma_1^{-4} (\partial_z^2 - 48\lambda_1) \left( \frac{5J_{\mathbf{a}}}{2g_2} z + \frac{\wp'}{\wp} \left( \frac{z}{\omega_{\mathbf{a}}} + \frac{\tau}{2}, \tau \right) \right) \right. \\ &\quad \left. + \gamma_2^{-4} (\partial_z^2 - 48\lambda_2) \left( \frac{5J_{\mathbf{a}}}{2g_2} z + \frac{\wp'}{\wp} \left( \frac{z}{\omega_{\mathbf{a}}}, \tau \right) \right) \right. \\ &\quad \left. + \gamma_3^{-4} (\partial_z^2 - 48\lambda_3) \left( \frac{5J_{\mathbf{a}}}{2g_2} z + \frac{\wp'}{\wp} \left( \frac{z}{\omega_{\mathbf{a}}} + \frac{1}{2}, \tau \right) \right) \right) + C_1, \end{aligned}$$

where  $C_0$  and  $C_1$  are some constants and  $\partial_z = d/dz$ .

*Proof.* Around  $\omega_j$  ( $j = 1, 2, 3$ ),

$$\frac{1}{P(\wp(z))} = \frac{1}{4}\gamma_j^{-2}(z - \omega_j)^{-2}(1 + O(z - \omega_j)^2)$$

since  $\wp(z) = \lambda_j + \gamma_j(z - \omega_j)^2 + O(z - \omega_j)^4$ . Hence

$$\frac{1}{P(\wp(z))} - \frac{1}{4}(\gamma_1^{-2}\wp(z - \omega_1) + \gamma_2^{-2}\wp(z - \omega_2) + \gamma_3^{-2}\wp(z - \omega_3)) \equiv \Gamma_0, \quad (5.1)$$

in which, by putting  $z = 0$ , we find  $\Gamma_0 = 9g_3(g_2^3 - 27g_3^2)^{-1}$ . We may set, for some  $c_0$ ,

$$\wp(z) + \frac{1}{\omega_{\mathbf{a}}}\partial_z \frac{\wp'}{\wp} \left( \frac{z}{\omega_{\mathbf{a}}} + \nu, \tau \right) \equiv c_0, \quad (5.2)$$

integration of which yields

$$\omega_{\mathbf{a}}c_0 = c_0 \int_{-\omega_1}^{\omega_1} dz = \int_{-\omega_1}^{\omega_1} \wp(z) dz = \int_{\mathbf{a}} \frac{\lambda}{\sqrt{P(\lambda)}} d\lambda.$$

Observing that

$$\begin{aligned} J_{\mathbf{a}} &= \int_{\mathbf{a}} \sqrt{P(\lambda)} d\lambda = \int_{\mathbf{a}} \left( \frac{2}{3} \lambda (\sqrt{P(\lambda)})' - \frac{2}{3} \frac{g_2 \lambda}{\sqrt{P(\lambda)}} - \frac{g_3}{\sqrt{P(\lambda)}} \right) d\lambda \\ &= -\frac{2}{3} J_{\mathbf{a}} - \frac{2}{3} g_2 \int_{\mathbf{a}} \frac{\lambda}{\sqrt{P(\lambda)}} d\lambda - g_3 \omega_{\mathbf{a}}, \end{aligned}$$

we have  $c_0 = -5J_{\mathbf{a}}/(2g_2\omega_{\mathbf{a}}) - 3g_3/(2g_2)$ . Inserting (5.2) with  $z \mapsto z - \omega_j$  ( $j = 1, 2, 3$ ) into (5.1), and using

$$\Gamma_0 - \frac{3g_3}{8g_2}(\gamma_1^{-2} + \gamma_2^{-2} + \gamma_3^{-2}) = 0,$$

we obtain the first primitive function. To derive the second formula, we note that, around  $z = \omega_j$  ( $j = 1, 2, 3$ ),

$$\begin{aligned} \frac{16}{P(\wp(z))^2} &= \left( \gamma_j^{-2} \wp(z - \omega_j) - 4\gamma_j^{-2} \lambda_j + O(z - \omega_j)^2 \right)^2 \\ &= \gamma_j^{-4} (\wp(z - \omega_j)^2 - 8\lambda_j \wp(z - \omega_j) + O(1)) \end{aligned}$$

by (5.1), since, say around  $z = \omega_1$ ,  $\wp(z - \omega_2) = \wp(\omega_3) + O(z - \omega_1)^2$ ,  $\wp(z - \omega_3) = \wp(\omega_2) + O(z - \omega_1)^2$ , with  $\gamma_2^{-2} \lambda_3 + \gamma_3^{-2} \lambda_2 + 4\Gamma_0 = -4\gamma_1^{-2} \lambda_1$ . Then we set

$$\begin{aligned} \frac{16}{P(\wp(z))^2} - \gamma_1^{-4} (\wp(z - \omega_1)^2 - 8\lambda_1 \wp(z - \omega_1)) - \gamma_2^{-4} (\wp(z - \omega_2)^2 - 8\lambda_2 \wp(z - \omega_2)) \\ - \gamma_3^{-4} (\wp(z - \omega_3)^2 - 8\lambda_3 \wp(z - \omega_3)) \equiv \Gamma_1 \end{aligned} \quad (5.3)$$

with

$$\Gamma_1 = \frac{7((\lambda_2 - \lambda_3)^4 \lambda_1^2 + (\lambda_3 - \lambda_1)^4 \lambda_2^2 + (\lambda_1 - \lambda_2)^4 \lambda_3^2)}{(\lambda_2 - \lambda_3)^4 (\lambda_3 - \lambda_1)^4 (\lambda_1 - \lambda_2)^4}.$$

Insertion of (5.2) and

$$\wp(z)^2 + \frac{\partial_z^3 \wp'}{6\omega_{\mathbf{a}} \wp} \left( \frac{z}{\omega_{\mathbf{a}}} + \nu, \tau \right) \equiv \frac{g_2}{12}$$

with  $z \mapsto z - \omega_j$  ( $j = 1, 2, 3$ ) into (5.3) leads to the second formula.  $\square$

**Lemma 5.2.** *Under the same supposition as in Lemma 5.1, for  $k = 0, 1, 2$ ,*

$$\begin{aligned} \int_{z_0}^z \frac{\wp(z)^k \wp'(z)}{P(\wp(z))} dz &= \frac{1}{4} \left( \gamma_1^{-1} \lambda_1^k \ln(\wp(z) - \lambda_1) \right. \\ &\quad \left. + \gamma_2^{-1} \lambda_2^k \ln(\wp(z) - \lambda_2) + \gamma_3^{-1} \lambda_3^k \ln(\wp(z) - \lambda_3) \right) + C(z_0), \end{aligned}$$

where  $z_0 \notin \{0, \omega_1, \omega_2, \omega_3\} + \omega_{\mathbf{a}}\mathbb{Z} + \omega_{\mathbf{b}}\mathbb{Z}$ , and  $C(z_0)$  is some constant.

*Proof.* Note that, for  $k = 0, 1, 2$ ,

$$\frac{4\wp^k}{P(\wp)} = \gamma_1^{-1} \lambda_1^k (\wp - \lambda_1)^{-1} + \gamma_2^{-1} \lambda_2^k (\wp - \lambda_2)^{-1} + \gamma_3^{-1} \lambda_3^k (\wp - \lambda_3)^{-1},$$

from which the lemma follows.  $\square$

Recall that  $\mathbf{p}(z) = \wp(z - t_0)$  and

$$\beta(z) = \frac{8e^{i\phi}}{5\omega_{\mathbf{a}}}(\beta_0 + \mathbf{b}(z - t_0)), \quad \mathbf{b}(\sigma) = -\frac{5}{4}e^{-i\phi}J_{\mathbf{a}}\sigma + \frac{\wp'}{\wp}\left(\frac{\sigma}{\omega_{\mathbf{a}}} + \nu, \tau\right). \quad (5.4)$$

Then we have the following.

**Lemma 5.3.** *Let  $\alpha_0 \in \mathbb{C}$  be a given number. Then*

$$\int_{\infty}^s (\mathbf{b}(\sigma - \alpha_0) + \mathbf{b}(\sigma + \alpha_0))_{\sigma} \mathbf{b}(\sigma) \frac{d\sigma}{\tilde{\sigma}} \ll s^{-1}, \quad \tilde{\sigma} = \sigma + t_0$$

as  $x = e^{i\phi}(\frac{5}{4}e^{-i\phi}s)^{4/5} \rightarrow \infty$  through  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ , in which the integral on the left-hand side is convergent.

*Proof.* Let  $x_{\nu} = e^{i\phi}(\frac{5}{4}e^{-i\phi}s_{\nu})^{4/5}$  be any sequence tending to  $\infty$  through  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ . Note that  $\mathbf{b}(s)$  and  $\mathbf{b}(s \pm \alpha_0)$  are bounded in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ . Integration by parts leads to

$$\begin{aligned} \int_{s_{\nu}}^s \mathbf{b}(\sigma - \alpha_0)_{\sigma} \mathbf{b}(\sigma) \frac{d\sigma}{\tilde{\sigma}} &= \left[ \mathbf{b}(\sigma - \alpha_0) \mathbf{b}(\sigma) \tilde{\sigma}^{-1} \right]_{s_{\nu}}^s \\ &\quad - \int_{s_{\nu}}^s \mathbf{b}(\sigma - \alpha_0) \mathbf{b}_{\sigma}(\sigma) \frac{d\sigma}{\tilde{\sigma}} + \int_{s_{\nu}}^s \mathbf{b}(\sigma - \alpha_0) \mathbf{b}(\sigma) \frac{d\sigma}{\tilde{\sigma}^2} \\ &= - \int_{s_{\nu} - \alpha_0}^{s - \alpha_0} \mathbf{b}(\sigma) \mathbf{b}_{\sigma}(\sigma + \alpha_0) \frac{d\sigma}{\tilde{\sigma} + \alpha_0} + O(s^{-1}) + O(s_{\nu}^{-1}) \\ &= - \int_{s_{\nu}}^s \mathbf{b}(\sigma + \alpha_0)_{\sigma} \mathbf{b}(\sigma) \frac{d\sigma}{\tilde{\sigma}} + O(s^{-1}) + O(s_{\nu}^{-1}). \end{aligned}$$

The passage to the limit  $s_{\nu} \rightarrow \infty$  leads to the lemma.  $\square$

By Lemma 5.1 with  $g_2 = -2e^{i\phi}$ , we may write

$$\begin{aligned} \frac{1}{P(\wp(z))} &= -\frac{1}{4\omega_{\mathbf{a}}} \frac{d}{dz} \left( \frac{\gamma_1^{-2}}{2} (\mathbf{b}(z - \omega_{\mathbf{a}}/2) + \mathbf{b}(z + \omega_{\mathbf{a}}/2)) \right. \\ &\quad \left. + \frac{\gamma_2^{-2}}{2} (\mathbf{b}(z - \omega_{\mathbf{a}}\nu) + \mathbf{b}(z + \omega_{\mathbf{a}}\nu)) + \frac{\gamma_3^{-2}}{2} (\mathbf{b}(z - \omega_{\mathbf{a}}\tau/2) + \mathbf{b}(z + \omega_{\mathbf{a}}\tau/2)) \right). \end{aligned}$$

Substituting  $z$  by  $z - t_0$  and using Lemma 5.3 we have the following.

**Lemma 5.4.** *In  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ ,*

$$\int_{\infty}^z \frac{\mathbf{b}(\zeta - t_0)}{P(\mathbf{p}(\zeta))} \frac{d\zeta}{\zeta} \ll z^{-1},$$

in which the integral on the left-hand is convergent.

**Lemma 5.5.** *In  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_{\infty}, c, \varepsilon)$ ,*

$$\int_{\infty}^z \frac{1}{P(\mathbf{p}(\zeta))} \frac{d\zeta}{\zeta} \ll z^{-1}, \quad \int_{\infty}^z \frac{1}{P(\mathbf{p}(\zeta))^2} \frac{d\zeta}{\zeta^2} = -\gamma_0 z^{-1} + O(z^{-2}), \quad \int_{\infty}^z \frac{\mathbf{p}(\zeta) \mathbf{p}'(\zeta)}{P(\mathbf{p}(\zeta))} \frac{d\zeta}{\zeta} \ll z^{-1},$$

where  $\gamma_0 = -5(8e^{3i\phi} + 27A_{\phi}^2)^{-1}$ .

*Proof.* Let  $\gamma_0 z + F(z)$  be the primitive function of  $1/P(\wp(z))^2$  given in Lemma 5.1, where  $F(z)$  is bounded in  $\check{\mathcal{D}}_{\text{cut}}(\phi, t_\infty, c, \varepsilon)$ . Then

$$\begin{aligned} \int_\infty^z \frac{1}{P(\mathfrak{p}(\zeta))^2} \frac{d\zeta}{\zeta^2} &= \left[ (F(\zeta - t_0) + \gamma_0 \zeta) \frac{1}{\zeta^2} \right]_\infty^z + 2 \int_\infty^z (F(\zeta - t_0) + \gamma_0 \zeta) \frac{d\zeta}{\zeta^3} \\ &= -\gamma_0 z^{-1} + O(z^{-2}), \end{aligned}$$

which is the second integral. The remaining estimates are similarly obtained by the use of Lemmas 5.1 and 5.2.  $\square$

## 6. DISCUSSION

Our main results Theorems 2.1 and 2.2 are derived by combining expressions (1.3) and (1.5) of  $y(x)$  and of  $B(\phi, t)$  in Theorems A and B, respectively, with analysis on the integral equations (associated with system (3.1)) in Proposition 3.3 via (4.3). The formulas of Theorems A and B with the error bounds  $O(t^{-\delta})$  have been obtained in [14] and [15] by WKB analysis on isomonodromy linear system (1.2). Our approach to the error term is said to be purely classical iterative calculation via (4.3) by the use of (1.3) and (1.5). This is just in the situation as referred to in [15, Remark 1]. The essential part of this procedure owes much to the structure of the integral relation (4.3) containing the term

$$\int_\infty^z F(\mathfrak{p}, \beta) \frac{d\zeta}{\zeta} \ll z^{-1},$$

in which the key to the estimate is Lemma 5.3. The corresponding error  $O(t^{-1})$ , say in [5], is a quantity arising in constructing the solution of the related Riemann-Hilbert problem.

It may be conjectured that the error bound of (1.5) in Theorem B is  $O(t^{-1})$ . If this is valid, then, in Theorem 2.2, the error bound of the expression for  $h(z)$  is  $O(z^{-2})$ . For the fifth Painlevé transcendents the corresponding error bound estimates are valid [19], [18].

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