ELLIPTIC ASYMPTOTIC REPRESENTATION OF THE FIFTH PAINLEVÉ TRANSCENDENTS (CORRECTED VERSION)

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ABSTRACT. For the fifth Painlevé transcendents an asymptotic representation by the Jacobi sn-function is presented in cheese-like strips along generic directions near the point at infinity. Its elliptic main part may be understood to depend on the phase shift as a single integration constant, which is parametrised by monodromy data for the associated isomonodromy deformation. The other integration constant is contained in the error term or in a correction function. This paper contains corrections of the Stokes graph and of the related results in the early version.

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1. INTRODUCTION

The fifth Painlevé equation

$$(\mathrm{P_V})\quad \frac{d^2y}{dx^2}=\Big(\frac{1}{2y}+\frac{1}{y-1}\Big)\Big(\frac{dy}{dx}\Big)^2-\frac{1}{x}\frac{dy}{dx}$$
$$+\frac{(y-1)^2}{8x^2}\Big((\theta_0-\theta_1+\theta_\infty)^2y-\frac{(\theta_0-\theta_1-\theta_\infty)^2}{y}\Big)+(1-\theta_0-\theta_1)\frac{y}{x}-\frac{y(y+1)}{2(y-1)}$$

with θ_0, θ_1, $\theta_\infty\in\mathbb{C}$ defines nonlinear special functions, which are meromorphic on the universal covering space of $\mathbb{C}\setminus\{0\}$. A general solution is expressed by a convergent series in a spiral domain around $x=0$, and admits asymptotic representations as $x\to\infty$ along the real and the imaginary axes (cf. e.g. [37], [36]). Using the isomonodromy property and the WKB analysis, for a generic case of $(\mathrm{P_V})$ Andreev and Kitaev [5] obtained families of solutions near $x=0$ and $x=\infty$ on the positive real axis, and connection formulas for these solutions. Along the imaginary axis asymptotic solutions and the associated monodromy data have been studied by [6], [38]. Furthermore Lisovyy et al. [29] gave a connection formula for the tau-function $\tau_{\mathrm{V}}(x)$ between $x=0$ and $x=i\infty$ and the ratios of multipliers of $\tau_{\mathrm{V}}(x)$ as $x\to 0,+\infty,i\infty$.

Near $x=\infty$ along directions other than the real or imaginary axis, a general solution of $(\mathrm{P_V})$ behaves quite differently. In generic directions it is known that, for solutions of the Painlevé equations $(\mathrm{P_I})$, ..., $(\mathrm{P_{IV}})$ except truncated or classical ones, the Boutroux ansatz holds [8], [9]. Elliptic asymptotic representations have been studied for $(\mathrm{P_I})$, $(\mathrm{P_{II}})$ by Joshi and Kruskal [20], [21], Its and Kapaev [15], Kapaev [22], [23], Kapaev and Kitaev [25], Kitaev [27], [28] and Novokshenov [30], [31]; for (P_{III}) by Novokshenov [32], [33]; and for (P_{IV}) by Kapaev [24], Vereshchagin [41]. The elliptic representations for (P_{II}) and (P_{III}), nonlinear Stokes phenomena and connection problems are also in the monograph [12] (see also [16]). Concerning the elliptic representation for solutions of (P_I) Iwaki's recent work [17] by the topological recursion is remarkable.

In this paper we show the Boutroux ansatz for the fifth Painlevé equation (P_V), that is, present an elliptic asymptotic representation for a general solution of (P_V), which is given by the Jacobi sn-function, along generic directions near the point at infinity (Theorem 2.1). In deriving our results we employ the isomonodromy property described as follows: for the complex parameter x the isomonodromy deformation of the two-dimensional linear system

$$(1.1)\qquad \frac{d\Xi}{d\xi} = \Big(\frac{x}{2}\sigma_3 + \frac{\mathcal{A}_0}{\xi} + \frac{\mathcal{A}_1}{\xi-1}\Big)\Xi, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathcal{A}_0 = \begin{pmatrix} \mathfrak{z}+\theta_0/2 & -u(\mathfrak{z}+\theta_0) \\ \mathfrak{z}/u & -\mathfrak{z}-\theta_0/2 \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} -\mathfrak{z}-(\theta_0+\theta_\infty)/2 & uy(\mathfrak{z}+(\theta_0-\theta_1+\theta_\infty)/2) \\ -(uy)^{-1}(\mathfrak{z}+(\theta_0+\theta_1+\theta_\infty)/2) & \mathfrak{z}+(\theta_0+\theta_\infty)/2 \end{pmatrix}$$

with $(y,\mathfrak{z},u) = (y(x),\mathfrak{z}(x),u(x))$ is governed by the system of equations

$$x\frac{dy}{dx} = xy - 2\mathfrak{z}(y-1)^2 - (y-1)\Big(\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)y - \frac{1}{2}(3\theta_0+\theta_1+\theta_\infty)\Big),$$

$$x\frac{d\mathfrak{z}}{dx} = y\mathfrak{z}\Big(\mathfrak{z}+\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)\Big) - \frac{(\mathfrak{z}+\theta_0)}{y}\Big(\mathfrak{z}+\frac{1}{2}(\theta_0+\theta_1+\theta_\infty)\Big),$$

$$x\frac{d}{dx}\log u = -2\mathfrak{z} - \theta_0 + y\Big(\mathfrak{z}+\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)\Big) + \frac{1}{y}\Big(\mathfrak{z}+\frac{1}{2}(\theta_0+\theta_1+\theta_\infty)\Big),$$

which is equivalent to (P_V) [19, Appendix C], [18], [5]; that is, $y(x)$ solves (P_V) if and only if the monodromy data M^0, M^1, S_1, S_2 for (1.1) (defined in Section 2) remain invariant under a small change of x. We apply the WKB analysis to calculate the monodromy data, which is used in deriving the elliptic expression of the main results. The main term expressed by the sn-function may be understood to depend on the phase shift only as a single integration constant parametrised by the monodromy data. The other integration constant is hidden in the error term, and is also deeply related to the correction function $B_\phi(t) = t(a_\phi(t) - A_\phi)$ with $\phi = \arg x$. The error term admits an explicit asymptotic formula containing the other integration constant [40, Theorems 2.2 and 2.3].

This paper is organised as follows. Section 2 describes our main results on the expression of a general solution by the Jacobi sn-function for $0 < |\phi| < \pi/2$ (Theorem 2.1) and for $0 < |\phi - \pi| < \pi/2$ (Theorem 2.2). Section 3 provides basic facts necessary in proving the main results: parametrisation of $y(x)$ by the monodromy data; turning points and Stokes curves for the symmetric linear system (3.4) with $\lambda = e^{i\phi}(2\xi - 1)$; and a WKB-solution in the canonical domain and local solutions around turning points. Section 4 is devoted to a direct monodromy problem for system (3.4). The monodromy matrices are obtained from connection matrices along suitably chosen paths, which are calculated by WKB analysis under the supposition (4.3). The monodromy matrices are expressed in term of integrals related to the characteristic roots and the WKB-solutions. In Section 5, these integrals are represented by elliptic integrals and the ϑ-function (Propositions 5.3, 5.5 and Corollary 5.4). In Section 6, considering an inverse monodromy problem by the use of the propositions in Section 5 and following the justification scheme by Kitaev [26], we arrive at the elliptic asymptotic representation of $y(x)$ as in Theorem 2.1 and an asymptotic expression of the correction function $B_{\phi}(t)$. For Theorem 2.2 the proof is sketched. Section 7 gives an alternative approach to $B_{\phi}(t)$, by using a system equivalent to $(\mathrm{P_V})$ via its Lagrangian. This system is employed in the study of the error term [40]. Several properties of the Boutroux equations and its solution are important in our arguments, which are discussed in the final section.

In the early version of [39] the Stokes graph is incorrect, and it affects the phase shifts of asymptotic solutions in [39, Theorems 2.1 and 2.2]. The amended Stokes graph and the subsequent modifications including the corrected phase shifts are contained in the second version (Corrigendum of [39]). In the preceding version, the further correction of the phase shift in Theorem 2.1 ($-\Omega_{\mathbf{a}}/2$ is added), and the following changes were made:

(i) Theorem 2.2 is renewed. Theorems 2.3 and 2.4 are replaced with unconditional results cited from [40], and the early proofs are removed from Section 6.

(ii) In the proof of Proposition 3.4 a minor revision is made. Estimates in Proposition 3.5 are improved. Remarks 3.7 and 4.1 are added, which are related to Proposition 3.5.

(iii) Subsection 4.2 is revised and Figure 4.2 is added.

(iv) The contents of Section 7 of the early version is replaced with a description on $B_{\phi}(t)$, which was contained in Section 6.

This version contains several minor revisions for readability, e.g. Theorem 2.1.

Throughout this paper we use the following symbols:

(1) σ_1, σ_2, σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

(2) for complex-valued functions f and g, we write $f \ll g$ or $g \gg f$ if $f = O(|g|)$, and write $f \asymp g$ if $g \ll f \ll g$.

2. Main results

To state our main theorems, we make necessary preparations. Let system (1.1) admit the isomonodromy property with respect to a fundamental matrix solution of the form

$$\Xi(\xi) = \Xi(x,\xi) = (I + O(\xi^{-1})) \exp\left(\frac{1}{2}(x\xi - \theta_{\infty}\log\xi)\sigma_3\right) \tag{2.1}$$

as $\xi \to \infty$ through the sector $|\arg(x\xi) - \pi/2| < \pi$. Let $M^0, M^1, M^{\infty} \in SL_2(\mathbb{C})$ be the monodromy matrices defined by $\Xi^{(l_\nu)}(\xi) = \Xi(\xi)M^{\nu}$ for $\nu = 0, 1, \infty$. Here $\Xi^{(l_\nu)}(\xi)$,

$\nu = 0, 1, \infty$ denote the analytic continuations of $\Xi(\xi)$ along the respective loops l_0, l_1, $l_\infty \in \pi_1(P^1(\mathbb{C}) \setminus \{0, 1, \infty\})$ as in Figure 2.1 defined for $-\pi/2 < \arg x < \pi/2$, which start from the point $p_{\rm st}$ satisfying $100 < |p_{\rm st}| < \infty$ and $\arg(xp_{\rm st}) = \pi/2$ and surround, respectively, $\xi = 0$, $\xi = 1$ and $\xi = \infty$ anticlockwise. It is easy to see $M^\infty M^1 M^0 = I$.

To define Stokes matrices, for each $k \in \mathbb{Z}$, denote by $\Xi_k(\xi)$ the matrix solution of

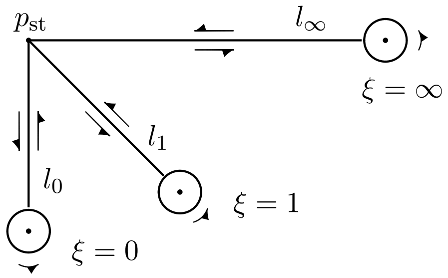


FIGURE 2.1. Loops l_0, l_1, l_∞ for $-\pi/2 < \arg x < \pi/2$

system (1.1) admitting the same asymptotic representation as (2.1) as $\xi \to \infty$ through the sector $|\arg(x\xi) - \pi/2 - (k-2)\pi| < \pi$. Then the Stokes matrices S_k are defined by $\Xi_{k+1}(\xi) = \Xi_k(\xi)S_k$, in particular, for $\Xi(\xi) = \Xi_2(\xi)$

$$\Xi(\xi) = \Xi_1(\xi)S_1, \quad \Xi_3(\xi) = \Xi(\xi)S_2, \quad S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

The monodromy data $M^0 = (m^0_{ij})$ and $M^1 = (m^1_{ij})$ satisfy (3.2) [5]. Set

$$\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)} := \{(M^0, M^1) \in SL_2(\mathbb{C})^2 \,|\, m^1_{11}m^0_{11} + m^1_{12}m^0_{21} = e^{-\pi i\theta_\infty}, \\ \mathrm{tr}\, M^0 = 2\cos\pi\theta_0,\ \mathrm{tr}\, M^1 = 2\cos\pi\theta_1\}/\sim,$$

where $(M^0, M^1) \sim (\tilde{M}^0, \tilde{M}^1)$ if $(M^0, M^1) = c^{\sigma_3}(\tilde{M}^0, \tilde{M}^1)c^{-\sigma_3}$ for some $c \in \mathbb{C} \setminus \{0\}$. Then $\dim_{\mathbb{C}} \mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)} = 2$. Let us call $\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$ the *monodromy manifold.* As shown in Proposition 3.1, for every $(M^0, M^1) \in \mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$ such that M^0, $M^1 \ne \pm I$, there exists uniquely a solution $y(x)$ of ($\mathrm{P_V}$) corresponding to (M^0, M^1), and then $y(x)$ is labelled with (M^0, M^1).

For $w(A, z) := \sqrt{(1-z^2)(A-z^2)}$ consider the Riemann surface $\Pi_A = \Pi_+ \cup \Pi_-$ such that Π_+ and Π_- are glued along the cuts $[-1, -A^{1/2}]$ and $[A^{1/2}, 1]$ with $\mathrm{Re}\, A^{1/2} \ge 0$. Let the branches be such that $z^{-2}\sqrt{(1-z^2)(A-z^2)} \to -1$ and $\sqrt{(A-z^2)/(1-z^2)} \to 1$ as $z \to \infty$ on the upper sheet Π_+.[1] Let $\mathbf{a}$ and $\mathbf{b}$ denote basic cycles on Π_A as drawn in Figure 2.2 when $A = A_\phi$. For a given number $\phi \in \mathbb{R}$, the Boutroux equations

$$\mathrm{Re}\, e^{i\phi} \int_{\mathbf{a}} \sqrt{\frac{A-z^2}{1-z^2}}dz = \mathrm{Re}\, e^{i\phi} \int_{\mathbf{b}} \sqrt{\frac{A-z^2}{1-z^2}}dz = 0 \tag{2.3}$$

admit a unique solution $A = A_\phi \in \mathbb{C}$ having the properties:

(i) $0 \le \mathrm{Re}\, A_\phi \le 1$ for $\phi \in \mathbb{R}$, and $A_0 = 0$, $A_{\pm\pi/2} = 1$;

(ii) $A_{-\phi} = \overline{A_\phi}$, $A_{\phi\pm\pi} = A_\phi$ for $\phi \in \mathbb{R}$;

[1] Then $A^{-1/2}\sqrt{(1-z^2)(A-z^2)} \to 1$ and $A^{-1/2}\sqrt{(A-z^2)/(1-z^2)} \to 1$ as $z \to 0$ on Π_+.

(iii) for $0 \le \phi \le \pi/2$, $\mathrm{Im}\, A_\phi \ge 0$, and, for $-\pi/2 \le \phi \le 0$, $\mathrm{Im}\, A_\phi \le 0$ (cf. Proposition 8.17 and Figure 8.2, (a)). By (i) we fix $\mathrm{Re}\, A_\phi^{1/2} \ge 0$, and then $(\overline{A_\phi})^{1/2} = \overline{A_\phi^{1/2}}$. In what follows let $\mathbf{a}$ and $\mathbf{b}$ be on $\Pi_{A_\phi} = \Pi_+ \cup \Pi_-$ as in Figure 2.2. Write

$$\Omega_{\mathbf{a},\mathbf{b}} = \Omega_{\mathbf{a},\mathbf{b}}(\phi) = \int_{\mathbf{a},\mathbf{b}} \frac{dz}{w(A_\phi, z)}, \quad \mathcal{E}_{\mathbf{a},\mathbf{b}} = \mathcal{E}_{\mathbf{a},\mathbf{b}}(\phi) = \int_{\mathbf{a},\mathbf{b}} \sqrt{\frac{A_\phi - z^2}{1-z^2}}\, dz,$$

and let $\mathrm{sn}(u;k)$ denote the Jacobi sn-function with modulus k.

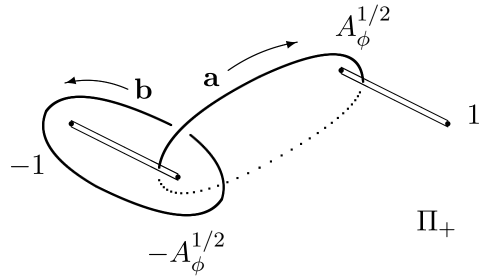


FIGURE 2.2. Cycles $\mathbf{a}$, $\mathbf{b}$ on Π_{A_ϕ}

2.1. **Elliptic representation.** For $0 < |\phi| < \pi/2$ we have the following.

Theorem 2.1. *Let $M^0 = (m^0_{ij})$ and $M^1 = (m^1_{ij})$ be such that $m^0_{11} m^0_{21} m^1_{12} \ne 0$ (respectively, $m^1_{11} m^0_{21} m^1_{12} \ne 0$). Then, for $-\pi/2 < \phi < 0$ (respectively, $0 < \phi < \pi/2$), a solution $y(x)$ labelled with (M^0, M^1) admits an elliptic representation of the form*

$$\frac{y(x)+1}{y(x)-1} = A_\phi^{1/2} \mathrm{sn}((x-x_0)/2 + \Delta(x); A_\phi^{1/2}),$$

with $\Delta(x) = O(x^{-\delta})$ as $x = e^{i\phi} t \to \infty$ through the cheese-like strip

$$S(\phi, t_\infty, \kappa_0, \delta_0) = \{x = e^{i\phi} t \mid \mathrm{Re}\, t > t_\infty,\ |\mathrm{Im}\, t| < \kappa_0\} \setminus \bigcup_{\rho \in \mathcal{P}_0} \{|x - \rho| < \delta_0\},$$

$$\mathcal{P}_0 = \{\rho \mid \mathrm{sn}((\rho - x_0)/2; A_\phi^{1/2}) = \infty\} = \{x_0 + \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}(2\mathbb{Z}+1)\},$$

$\delta > 0$ being some positive number, $\kappa_0 > 0$ a given number, $\delta_0 > 0$ a given small number, and $t_\infty = t_\infty(\kappa_0, \delta_0)$ a large number depending on (κ_0, δ_0). Furthermore x_0 is such that

$$\begin{aligned} x_0 &\equiv \frac{-1}{\pi i}\Big(\Omega_{\mathbf{b}} \log(m^0_{21} m^1_{12}) + \Omega_{\mathbf{a}} \log \mathfrak{m}_\phi\Big) - \Big(\frac{\Omega_{\mathbf{a}}}{2} + \Omega_{\mathbf{b}}\Big)(\theta_\infty + 1) - \frac{\Omega_{\mathbf{a}}}{2} \\ &= \frac{-1}{\pi i}\Big(\Omega_{\mathbf{b}} \log(e^{\pi i \theta_\infty} m^0_{21} m^1_{12}) + \Omega_{\mathbf{a}} \log \mathfrak{m}_{\phi,\theta_\infty}\Big) - \Omega_{\mathbf{a}} - \Omega_{\mathbf{b}} \mod 2\Omega_{\mathbf{a}}\mathbb{Z} + 2\Omega_{\mathbf{b}}\mathbb{Z}, \end{aligned}$$

where $\mathfrak{m}_{\phi,\theta_\infty} = e^{\pi i \theta_\infty/2} m^0_{11}$ if $-\pi/2 < \phi < 0$, and $= e^{-\pi i \theta_\infty/2}(m^1_{11})^{-1}$ if $0 < \phi < \pi/2$, and $\mathfrak{m}_\phi = e^{-\pi i \theta_\infty/2} \mathfrak{m}_{\phi,\theta_\infty}$.

Remark 2.1. In Theorem 2.1 the complementary cases, say $m^1_{11} m^0_{21} m^1_{12} = 0$ for $0 < \phi < \pi/2$ correspond to truncated or triply-truncated solutions [5, Corollaries 5.2 and 5.3], [4, §5], [2, Theorem 1.1], [38, Theorem 2.21] (see also Remark 3.5). In Theorem 2.2 below similar truncated solutions exist in the complementary cases.

Remark 2.2. The pair $(\mathfrak{m}_\phi, m^0_{21}m^1_{12})$ uniquely determines $(M^0, M^1) \in \mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$.

Remark 2.3. The point $x = \rho$ is in $\mathcal{P}_0$ if and only if $y(\rho) = 1$.

Remark 2.4. Our calculation of Section 4 leads to $\delta = 1/4 - \varepsilon$ for any $0 < \varepsilon < 1/4$, whose numerical value is caused by a technical reason. In fact we have $\delta = 1$ in Theorem 2.3, [40, Theorem 2.1].

The solution in Theorem 2.1 labelled with (M^0, M^1) may be written as $y(x, M^0, M^1)$. Let us denote by the same symbol $y(x, M^0, M^1)$ the analytic continuation to the sectors $0 < |\phi - \pi| < \pi/2$. Note that $\Omega_{\mathbf{a}}$ and $\Omega_{\mathbf{b}}$ are determined by A_ϕ, which does not depend on M^0, M^1, and satisfy $\Omega_{\mathbf{a},\mathbf{b}}(\phi + \pi) = \Omega_{\mathbf{a},\mathbf{b}}(\phi)$.

Theorem 2.2. *Write* $\breve{M}^0 = (\breve{m}^0_{ij}) = S_2^{-1}M^0S_2$ *and* $\breve{M}^1 = (\breve{m}^1_{ij}) = S_2^{-1}M^1S_2$. *Suppose that* $\breve{m}^0_{22}\breve{m}^0_{12}\breve{m}^1_{21} \ne 0$ *(respectively,* $\breve{m}^1_{22}\breve{m}^0_{12}\breve{m}^1_{21} \ne 0$*). Then, for* $\pi/2 < \phi < \pi$ *(respectively,* $\pi < \phi < 3\pi/2$*),* $y(x) = y(x, M^0, M^1)$ *admits an elliptic representation of the form*

$$\frac{y(x)+1}{y(x)-1} = A_\phi^{1/2}\mathrm{sn}((x - \breve{x}_0)/2 + \Delta(x); A_\phi^{1/2}),$$

as $x = e^{i\phi}t \to \infty$ *through* $S(\phi, t_\infty, \kappa_0, \delta_0)$*, where the phase shift is given by*

$$\begin{aligned}\breve{x}_0 &\equiv \frac{-1}{\pi i}\Big(\Omega_{\mathbf{b}}\log(\breve{m}^0_{12}\breve{m}^1_{21}) + \Omega_{\mathbf{a}}\log\breve{\mathfrak{m}}_\phi\Big) - \Big(\frac{\Omega_{\mathbf{a}}}{2} + \Omega_{\mathbf{b}}\Big)(\theta_\infty + 1) - \frac{\Omega_{\mathbf{a}}}{2}\\ &= \frac{-1}{\pi i}\Big(\Omega_{\mathbf{b}}\log(e^{\pi i\theta_\infty}(\breve{m}^0_{12}\breve{m}^1_{21})^{-1}) + \Omega_{\mathbf{a}}\log\breve{\mathfrak{m}}_{\phi,\theta_\infty}\Big) - \Omega_{\mathbf{a}} - \Omega_{\mathbf{b}} \mod 2\Omega_{\mathbf{a}}\mathbb{Z} + 2\Omega_{\mathbf{b}}\mathbb{Z}\end{aligned}$$

with $\breve{\mathfrak{m}}_{\phi,\theta_\infty} = e^{\pi i\theta_\infty/2}(\breve{m}^0_{22})^{-1}$ *if* $\pi/2 < \phi < \pi$*, and* $= e^{-\pi i\theta_\infty/2}\breve{m}^1_{22}$ *if* $\pi < \phi < 3\pi/2$*, and* $\breve{\mathfrak{m}}_\phi = e^{-\pi i\theta_\infty/2}\breve{\mathfrak{m}}_{\phi,\theta_\infty}$.

Remark 2.5. Let $0 < |\phi - 2p\pi| < \pi/2$ or $0 < |\phi - 2p\pi - \pi| < \pi/2$ with $p \in \mathbb{Z} \setminus \{0\}$. Set

$$\begin{aligned}M^0_p &= ((m^0_p)_{ij}) = U_p^{-1}M^0U_p, \qquad & M^1_p &= ((m^1_p)_{ij}) = U_p^{-1}M^1U_p,\\ \breve{M}^0_p &= ((\breve{m}^0_p)_{ij}) = \breve{U}_p^{-1}M^0\breve{U}_p, \qquad & \breve{M}^1_p &= ((\breve{m}^1_p)_{ij}) = \breve{U}_p^{-1}M^1\breve{U}_p,\end{aligned}$$

where

$$U_p = \begin{cases} S_2S_3\cdots S_{2p}S_{2p+1}, & \text{if } p > 0,\\ I, & \text{if } p = 0,\\ S_1^{-1}S_0^{-1}\cdots S_{2p+3}^{-1}S_{2p+2}^{-1}, & \text{if } p < 0,\end{cases} \qquad \breve{U}_p = U_pS_{2p+2} \quad \text{for } p \in \mathbb{Z}.$$

Since $S_{k+2} = e^{i\pi\theta_\infty\sigma_3}S_ke^{-i\pi\theta_\infty\sigma_3}$ for $k \in \mathbb{Z}$ [5, (2.5)], we have $U_p = (M^1M^0)^{-p}e^{-\pi i\theta_\infty p\sigma_3}$. Then $y(M_0, M_1; x)$ admits an elliptic representation as of Theorem 2.1 or 2.2 with the following substitutions in the phase shift (see Section 6.2):

(1) for $0 < |\phi - 2p\pi| < \pi/2$, $m^0_{21}m^1_{12} \mapsto (m^0_p)_{21}(m^1_p)_{12}$, $\mathfrak{m}_\phi \mapsto \mathfrak{m}^p_\phi = \mathfrak{m}_\phi|_{m^{0,1}_{11} \mapsto (m^{0,1}_p)_{11}}$ in Theorem 2.1; and

(2) for $0 < |\phi - 2p\pi - \pi| < \pi/2$, $(\breve{m}^0_{12}\breve{m}^1_{21})^{-1} \mapsto ((\breve{m}^0_p)_{12}(\breve{m}^1_p)_{21})^{-1}$, $\breve{\mathfrak{m}}_\phi \mapsto \breve{\mathfrak{m}}^p_\phi = \breve{\mathfrak{m}}_\phi|_{\breve{m}^{0,1}_{22} \mapsto (\breve{m}^{0,1}_p)_{22}}$ in Theorem 2.2.

2.2. **Error term $\Delta(x)$.** The elliptic expression above apparently contains the single integration constant x_0, and the other one is hidden in the error term $\Delta(x) \ll x^{-\delta}$. Let us define necessary functions and constants to express $\Delta(x)$.

For $y(x) = y(x, M^0, M^1)$ of Theorem 2.1, let $b(x)$ be such that

$$a_\phi = A_\phi + \frac{B_\phi(t)}{t} = A_\phi + \frac{b(x)}{x} \quad \text{with } x = e^{i\phi}t, \quad \text{i.e. } b(e^{i\phi}t) = e^{i\phi}B_\phi(t),$$

which is related to $y(x)$ via (3.10), (7.5) with $y^* = y'(t)$. Set

$$\begin{aligned}
\psi_0(x) &= A_\phi^{1/2}\mathrm{sn}((x - x_0)/2; A_\phi^{1/2}),\\
b_0(x) &= \beta_0 - \frac{2\mathcal{E}_{\mathbf{a}}}{\Omega_{\mathbf{a}}}x - \frac{8}{\Omega_{\mathbf{a}}}\frac{\vartheta'}{\vartheta}\Big(\frac{1}{2\Omega_{\mathbf{a}}}(x - x_0), \tau_0\Big),\\
\tau_0 &= \frac{\Omega_{\mathbf{b}}}{\Omega_{\mathbf{a}}}, \qquad \beta_0 = -\frac{8}{\Omega_{\mathbf{a}}}(\log(m^0_{21}m^1_{12}) + \pi i(\theta_\infty + 1)),
\end{aligned}$$

where

$$\vartheta(z, \tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi i z n}, \qquad \mathrm{Im}\,\tau > 0$$

with $\vartheta'(z, \tau) = (d/dz)\vartheta(z, \tau)$ is the ϑ-function (cf. Section 5.2). Then $\psi_0(x)$ solves $2\psi_0' = w(A_\phi, \psi_0) = \sqrt{(1 - \psi_0^2)(A_\phi - \psi_0^2)}$, and $b_0(x)$ fulfills $b_0(x) - b(x) = b_0(e^{i\phi}t) - e^{i\phi}B_\phi(t) \ll t^{-\delta}$ in $S(\phi, t_\infty, \kappa_0, \delta_0)$ (Proposition 5.6 and Corollary 6.1). Furthermore as in Section 7, $b_0'(x) = 2(\psi_0(x)^2 - A_\phi) + 4\psi_0'(x)$. Write

$$\begin{aligned}
\check{S}(\phi, t_\infty, \kappa_0, \delta_0) &= S(\phi, t_\infty, \kappa_0, \delta_0) \setminus \bigcup_{\rho\in\mathcal{Q}} \{|x - \rho| < \delta_0\},\\
\mathcal{Q} &= \{\rho \mid \mathrm{sn}((\rho - x_0)/2; A_\phi^{1/2}) = \pm A_\phi^{-1/2},\ \pm 1\}.
\end{aligned}$$

For $\sigma = e^{i\phi}t_\sigma \in \mathcal{Q}$ let $l(\sigma)$ be the line defined by $x = e^{i\phi}(\mathrm{Re}\,t_\sigma + i\eta)$ with $\eta \ge \mathrm{Im}\,t_\sigma$ if $\mathrm{Im}\,t_\sigma \ge 0$ (respectively, $\eta < \mathrm{Im}\,t_\sigma$ if $\mathrm{Im}\,t_\sigma < 0$); and, if necessary, modify $l(\sigma)$ not to touch other circles $|x - \sigma'| = \delta_0$ with $\sigma' \in \mathcal{P}_0 \cup \mathcal{Q} \setminus \{\sigma\}$. Then $\check{S}_{\mathrm{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$ denote $\check{S}(\phi, t_\infty, \kappa_0, \delta_0)$ equipped with the cuts along $l(\sigma)$ or its modification for all $\sigma \in \mathcal{Q}$.

For $\Delta(x)$ and $b(x) - b_0(x)$ we have the following [40, Theorems 2.2 and 2.3].

Theorem 2.3. *The error term $\Delta(x) = h(x)/2$ is explicitly represented by*

$$h(x) = -\frac{2((\theta_0 - \theta_1)^2 + \theta_\infty^2)}{A_\phi - 1}x^{-1} - \int_\infty^x F_1(\psi_0, b_0)\frac{d\xi}{\xi} - \frac{3}{2}\int_\infty^x F_1(\psi_0, b_0)^2\frac{d\xi}{\xi^2} + O(x^{-2}),$$

with

$$F_1(\psi_0, b_0) = \frac{4(\theta_0 + \theta_1)\psi_0(\xi) - b_0(\xi)}{2(A_\phi - \psi_0(\xi)^2)}.$$

Here

$$\int_\infty^x F_1(\psi_0, b_0)\frac{d\xi}{\xi} \ll x^{-1}, \qquad \int_\infty^x F_1(\psi_0, b_0)^2\frac{d\xi}{\xi^2} \ll x^{-1}$$

as $x \to \infty$ through $\check{S}_{\mathrm{cut}}(\phi, t_\infty, \kappa_0, \delta_0)$. Furthermore,

$$xh(x) = h_0\beta_0^2 + h_1(x)\beta_0 + h_2(x) + O(x^{-1}),$$

where $h_0 = (1/8)A_\phi^{-1}(1 - A_\phi)^{-1}$, $h_1(x) \ll x^{-1}$ and $h_2(x) \ll x^{-1}$.

Theorem 2.4. *We have*

$$b(x)-b_0(x)=b_0'(x)h(x)$$
$$-4((\theta_0-\theta_1)^2+\theta_\infty^2)x^{-1}-\int_\infty^x (A_\phi-\psi_0^2)F_1(\psi_0,b_0)^2\frac{d\xi}{\xi^2}+O(x^{-2}),$$

in which $b_0'(x)=4\psi_0'-2(A_\phi-\psi_0^2)$, *and*

$$\int_\infty^x (A_\phi-\psi_0^2)F_1(\psi_0,b_0)^2\frac{d\xi}{\xi^2}\ll x^{-1}$$

as $x\to\infty$ *through* $\check{S}_{\mathrm{cut}}(\phi,t_\infty,\kappa_0,\delta_0)$, *and* $b(x)-b_0(x)\ll x^{-1}$ *in* $S(\phi,t_\infty,\kappa_0,\delta_0)$.

By Theorem 2.3 we also have

$$\frac{y(x)+1}{y(x)-1}=A_\phi^{1/2}\mathrm{sn}((x-x_0)/2;A_\phi^{1/2})+O(x^{-1})$$

in $S(\phi,t_\infty,\kappa_0,\delta_0)$ [40, Theorem 2.1].

3. Basic facts

3.1. **Parametrisation of $y(x)$ by the monodromy data.** Note that

$$M^1M^0=S_1^{-1}e^{-\pi i\theta_\infty\sigma_3}S_2^{-1} \tag{3.1}$$

[5, (2.8), (2.13)]. For the monodromy matrices $M^0=(m^0_{ij})$, $M^1=(m^1_{ij})$, let $\mathcal{M}^*_{(\theta_0,\theta_1,\theta_\infty)}$ be the algebraic variety consisting of $(M^0,M^1)\in SL_2(\mathbb{C})^2$ such that

$$\begin{aligned}&\mathrm{tr}\,M^0=2\cos\pi\theta_0,\quad \mathrm{tr}\,M^1=2\cos\pi\theta_1,\\ &(M^1M^0)_{11}=m^1_{11}m^0_{11}+m^1_{12}m^0_{21}=e^{-\pi i\theta_\infty},\end{aligned} \tag{3.2}$$

which is called the *manifold of monodromy data* in [5]. Then $\dim_{\mathbb{C}}\mathcal{M}^*_{(\theta_0,\theta_1,\theta_\infty)}=3$. Our monodromy manifold $\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$ defined in Section 2 is written as $\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}=\mathcal{M}^*_{(\theta_0,\theta_1,\theta_\infty)}/\sim$. For $d_0\in\mathbb{C}\setminus\{0\}$, the gauge transformation $\Xi=d_0^{\sigma_3}\hat{\Xi}$ changes (1.1) with $(y,\mathfrak{z},u)$ to an isomonodromy system with $(y,\mathfrak{z},d_0^{-2}u)$, and the monodromy matrices for the canonical solution $\hat{\Xi}(\xi)$ become $d_0^{-\sigma_3}M^0d_0^{\sigma_3}$, $d_0^{-\sigma_3}M^1d_0^{\sigma_3}$. By this fact combined with the surjectivity of the Riemann-Hilbert correspondence [5, §§3, 4, 5], [2, 3, 4, 7, 34], and the uniqueness [5, Propositions 2.1 and 2.2] (see also Proposition 3.2, [12, Proposition 5.9 and Theorem 5.5], [13]) we have the following.

Proposition 3.1. *Let* $\mathcal{Y}(\mathrm{P_V})$ *be the family of solutions of* $(\mathrm{P_V})$, *and let* $\varphi:\mathcal{Y}(\mathrm{P_V})\to\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$ *be such that* $y\mapsto(M^0,M^1)$ *if* $(y,\mathfrak{z},u)$ *governs the isomonodromy deformation of* (1.1) *with monodromy data* (M^0,M^1). *Then we have the canonical bijection*

$$\varphi:\ \mathcal{Y}(\mathrm{P_V})\setminus\mathcal{Y}_0(\mathrm{P_V})\to\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}\setminus\mathcal{M}_0,$$

where $\mathcal{M}_0=\{(M^0,M^1)\in\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}\mid M^0=\pm I$ *or* $M^1=\pm I\}$, $\mathcal{Y}_0(\mathrm{P_V})=\varphi^{-1}(\mathcal{M}_0)$.

Thus the solutions in $\mathcal{Y}(\mathrm{P_V})\setminus\mathcal{Y}_0(\mathrm{P_V})$ are parametrised by $(M^0,M^1)\in\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}\setminus\mathcal{M}_0$, essentially two parameters.

Remark 3.1. If $\theta_0 \in \mathbb{Z}$ or $\theta_1 \in \mathbb{Z}$, then there exists a one-parameter family of classical solutions corresponding to $M^0 = I$ or $M^1 = I$, which is represented by the Whittaker function. In particular, if $\theta_0 = 0$ and $\mathfrak{z} = 0$, then $\mathcal{A}_0$ in (1.1) vanishes and $(\mathrm{P_V})$ admits a family of solutions solving

$$x\frac{dy}{dx} = xy - \frac{1}{2}(y-1)(\theta_\infty(y-1) - \theta_1(y+1))$$

[5, Remark 2.1].

3.2. **Symmetric linear system.** To consider $y(x) \in \mathcal{Y}(\mathrm{P_V})$ along a ray $\arg x = \phi$ with $|\phi| < \pi/2$, and to convert (1.1) to a symmetric form, set

$$x = e^{i\phi}t, \quad t > 0, \qquad \xi = (e^{-i\phi}\lambda + 1)/2. \tag{3.3}$$

Then by the gauge transformation $Y = \exp(-\varpi(t,\phi)\sigma_3)\Xi$ with $\varpi(t,\phi) = e^{i\phi}t/4 + (\theta_\infty/2)(i\phi + \log 2)$ system (1.1) is taken to

$$\frac{dY}{d\lambda} = t\mathcal{B}(t,\lambda)Y \tag{3.4}$$

with $\mathcal{B}(t,\lambda) = \hat{u}^{\sigma_3/2}(b_3\sigma_3 + b_2\sigma_2 + b_1\sigma_1)\hat{u}^{-\sigma_3/2}$. Here $\hat{u} = u\exp(-2\varpi(t,\phi))$ and

$$\begin{aligned} b_3 = b_3(t,\lambda) &= \frac{1}{4} + t^{-1}\Big(\frac{a_0^{11}}{\lambda + e^{i\phi}} + \frac{a_1^{11}}{\lambda - e^{i\phi}}\Big), \\ b_2 = b_2(t,\lambda) &= \frac{it^{-1}}{2}\Big(\frac{a_0^{12} - a_0^{21}}{\lambda + e^{i\phi}} + \frac{a_1^{12} - a_1^{21}}{\lambda - e^{i\phi}}\Big), \\ b_1 = b_1(t,\lambda) &= \frac{t^{-1}}{2}\Big(\frac{a_0^{12} + a_0^{21}}{\lambda + e^{i\phi}} + \frac{a_1^{12} + a_1^{21}}{\lambda - e^{i\phi}}\Big), \end{aligned} \tag{3.5}$$

a_0^{ij} and a_1^{ij} being the entries of $\mathcal{A}_0|_{u=1}$, $\mathcal{A}_1|_{u=1}$, that is,

$$\begin{aligned} a_0^{11} &= \mathfrak{z} + \frac{\theta_0}{2}, & a_0^{12} &= -(\mathfrak{z} + \theta_0), \\ a_0^{21} &= \mathfrak{z}, & a_0^{22} &= -a_0^{11}; \\ a_1^{11} &= -\mathfrak{z} - \frac{\theta_0 + \theta_\infty}{2}, & a_1^{12} &= y\Big(\mathfrak{z} + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\Big), \\ a_1^{21} &= -\frac{1}{y}\Big(\mathfrak{z} + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\Big), & a_1^{22} &= -a_1^{11}. \end{aligned}$$

Note that

$$\begin{aligned} &(\lambda^2 - e^{2i\phi})b_3 = \frac{1}{4}(\lambda^2 - e^{2i\phi}) - 2e^{i\phi}t^{-1}\mathfrak{z} - \frac{1}{2}(\theta_\infty\lambda + (2\theta_0 + \theta_\infty)e^{i\phi})t^{-1}, \\ &y(\lambda^2 - e^{2i\phi})(b_1 + ib_2) = ((y-1)(\lambda + e^{i\phi}) - 2e^{i\phi}y)t^{-1}\mathfrak{z} - \frac{1}{2}(\theta_0 + \theta_1 + \theta_\infty)(\lambda + e^{i\phi})t^{-1}, \\ &(\lambda^2 - e^{2i\phi})(b_1 - ib_2) = ((y-1)(\lambda + e^{i\phi}) + 2e^{i\phi})t^{-1}\mathfrak{z} \\ &\qquad\qquad + \Big(\frac{y}{2}(\theta_0 - \theta_1 + \theta_\infty)(\lambda + e^{i\phi}) - \theta_0(\lambda - e^{i\phi})\Big)t^{-1}. \end{aligned}$$

Let loops $\hat{l}_0$, $\hat{l}_1$ and a point $\hat{p}_{\mathrm{st}}$ in the λ-plane be the images of l_0, l_1 and p_{st} under (3.3). The loops $\hat{l}_0$, $\hat{l}_1$ start from $\hat{p}_{\mathrm{st}}$ and surround $\lambda = -e^{i\phi}$, $\lambda = e^{i\phi}$, respectively; and $\arg \hat{p}_{\mathrm{st}} = \pi/2$ (cf. Figure 3.1).

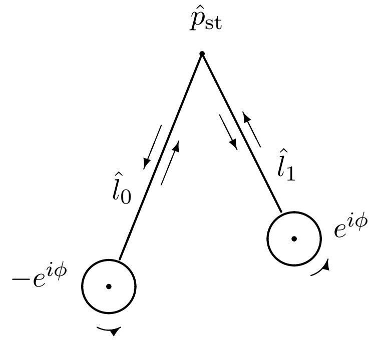


FIGURE 3.1. Loops $\hat{l}_0$ and $\hat{l}_1$ on the λ-plane

Then (3.4) admits the matrix solution $Y(t,\lambda)=\exp(-\varpi(t,\phi)\sigma_3)\Xi(e^{i\phi}t,(e^{-i\phi}\lambda+1)/2)$ (cf. (2.1)) with the properties:

(i) $Y(t,\lambda)$ has the asymptotic representation

$$Y(t,\lambda)=(I+O(\lambda^{-1}))\exp\Big(\frac{1}{4}(t\lambda-2\theta_\infty\log\lambda)\sigma_3\Big) \tag{3.6}$$

as $\lambda\to\infty$ through the sector $|\arg\lambda-\pi/2|<\pi$, the branch of $\log\lambda$ being taken in such a way that $\mathrm{Im}\,(\log\lambda)\to\pi/2$ as $\lambda\to\infty$ through this sector;

(ii) the isomonodromy deformation yields the same monodromy data M^0, M^1, S_1, S_2 as in Section 2, where M^0 and M^1 are defined by $Y^{(\hat{l}_\nu)}(t,\lambda)=Y(t,\lambda)M^\nu$ for $\nu=0,1$ with $Y^{(\hat{l}_\nu)}(t,\lambda)$ denoting the analytic continuation of $Y(t,\lambda)$ along the loop $\hat{l}_\nu$, and S_k are such that $Y_{k+1}(t,\lambda)=Y_k(t,\lambda)S_k$ with $Y_k(t,\lambda)$ $(Y_2(t,\lambda)=Y(t,\lambda))$ having the same asymptotic representation as (3.6) in the sector $|\arg\lambda-\pi/2-(k-2)\pi|<\pi$;

(iii) system (3.4) has the isomonodromy property if and only if $(y,\mathfrak{z},\hat{u})$ with $\hat{u}=u\exp(-2\varpi(t,\phi))$ satisfies

$$\begin{aligned}
&ty_t=e^{i\phi}ty-2\mathfrak{z}(y-1)^2-\frac{(y-1)}{2}\Big((\theta_0-\theta_1+\theta_\infty)y-(3\theta_0+\theta_1+\theta_\infty)\Big),\\
&t\mathfrak{z}_t=y\mathfrak{z}\Big(\mathfrak{z}+\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)\Big)-\frac{1}{y}(\mathfrak{z}+\theta_0)\Big(\mathfrak{z}+\frac{1}{2}(\theta_0+\theta_1+\theta_\infty)\Big),\\
&\frac{t\hat{u}_t}{\hat{u}}=-2\mathfrak{z}-\theta_0+y\Big(\mathfrak{z}+\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)\Big)+\frac{1}{y}\Big(\mathfrak{z}+\frac{1}{2}(\theta_0+\theta_1+\theta_\infty)\Big)
\end{aligned} \tag{3.7}$$

$(y_t=dy/dt)$, and then $y(e^{i\phi}t)$ is parametrised by $(M^0,M^1)\in\mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$.

Remark 3.2. In what follows we denote $y(e^{i\phi}t)$ by $y(t)$ for brevity, and set

$$\mathfrak{z}=-\frac{(y_t-e^{i\phi}y)t}{2(y-1)^2}-\frac{1}{4}(\theta_0-\theta_1+\theta_\infty)+\frac{\theta_0+\theta_1}{2(y-1)}, \tag{3.8}$$

which is the first equation of (3.7).

Remark 3.3. Let $\mathbf{s}$ be a substitution given by $e^{i\phi}\mapsto -e^{i\phi}$, $y\mapsto y^{-1}$, $(\theta_0,\theta_1)\mapsto(\theta_1,\theta_0)$. It is easy to see that $\mathbf{s}(\mathfrak{z})=-\mathfrak{z}-(\theta_0+\theta_1+\theta_\infty)/2$. Then system (3.4) is invariant under the extension of $\mathbf{s}$: $(\mathbf{s},Y\mapsto y^{\sigma_3/2}Y)$.

The uniqueness of the Riemann-Hilbert correspondence for system (3.4) will be used in the justification procedure in Section 6.

Proposition 3.2. *If $M^0 \neq I$ and $M^1 \neq I$, a canonical solution $Y_2(t,\lambda)$ of (3.4) corresponding to (M^0, M^1) is uniquely determined.*

Proof. Let $Y_2(\lambda)$ and $\tilde{Y}_2(\lambda)$ be canonical solutions defining (M^0, M^1). Suppose that $Y_2(\lambda)$ and $\tilde{Y}_2(\lambda)$ admit the same asymptotic representation (3.6) as $\lambda \to \infty$ in the sector $|\arg\lambda - \pi/2| < \pi$, and set, around $\lambda = -e^{i\phi}$, i.e., $\lambda_- := \lambda + e^{i\phi} = 0$,

$$Y_2(\lambda) = G_0(I + O(\lambda_-))\lambda_-^{(\theta_0/2)\sigma_3 + L} C_0, \quad \tilde{Y}_2(\lambda) = \tilde{G}_0(I + O(\lambda_-))\lambda_-^{(\theta_0/2)\sigma_3 + \tilde{L}} \tilde{C}_0,$$

where G_0, $\tilde{G}_0$, C_0, $\tilde{C}_0 \in SL_2(\mathbb{C})$ and $L = (l_{ij})$ and $\tilde{L} = (\tilde{l}_{ij})$ are upper-triangular (respectively, lower-triangular) matrices with vanishing diagonal entries. Then we have $M^0 = C_0^{-1} e^{\pi i\theta_0\sigma_3} e^{2\pi i L} C_0 = \tilde{C}_0^{-1} e^{\pi i\theta_0\sigma_3} e^{2\pi i\tilde{L}} \tilde{C}_0$, and hence $M^0 \neq \pm I$ implies either of the cases: (i) $\theta_0 \notin \mathbb{Z}$ and $L = \tilde{L} = 0$; (ii) $\theta_0 \in \mathbb{Z}_{\geq 0}$ and $l_{12}\tilde{l}_{12} \neq 0$ (respectively, $\theta_0 \in \mathbb{Z}_{\leq 0}$ and $l_{21}\tilde{l}_{21} \neq 0$). The relation $(C_0\tilde{C}_0^{-1})^{-1} e^{\pi i\theta_0\sigma_3} e^{2\pi i L} C_0 \tilde{C}_0^{-1} = e^{\pi i\theta_0\sigma_3} e^{2\pi i\tilde{L}}$ leads to $C_0\tilde{C}_0^{-1} = \mathrm{diag}[\alpha, \alpha^{-1}]$ or $\mathrm{diag}[\alpha, \alpha^{-1}] + L_*$ with some $\alpha \neq 0$ and some upper-triangular (respectively, lower-triangular) matrix L_* with vanishing diagonal entries. By this fact

$$Y_2(\lambda)\tilde{Y}_2(\lambda)^{-1} = G_0(I + O(\lambda_-))\lambda_-^{(\theta_0/2)\sigma_3} e^{2\pi i L} C_0 \tilde{C}_0^{-1} e^{-2\pi i\tilde{L}} \lambda_-^{-(\theta_0/2)\sigma_3} (I + (\lambda_-))\tilde{G}_0^{-1}$$

is holomorphic around $\lambda = -e^{i\phi}$. Similarly $Y_2(\lambda)\tilde{Y}_2(\lambda)^{-1}$ is also holomorphic around $\lambda = e^{i\phi}$ if $M^1 \neq \pm I$. Observing that $Y_2(\lambda)\tilde{Y}_2(\lambda)^{-1} = I + O(\lambda^{-1})$ as $\lambda \to \infty$ through $|\arg\lambda - \pi/2| < \pi$, we conclude $Y_2(\lambda) = \tilde{Y}_2(\lambda)$ by the Phragmén-Lindelöf reasoning. □

3.3. **Characteristic roots, turning points and Stokes curves.** In the remaining part of this section, Sections 4 and 5 we are concerned with the direct monodromy problem for system (3.4), that is, calculation of the monodromy. Then y, $\mathfrak{z}$ and u may be treated as arbitrary complex parameters. It is also possible, in place of $\mathfrak{z}$, to choose $y^* = y_t$ related to $\mathfrak{z}$ via (3.8), and then let us suppose that y, y^* and u are arbitrary complex parameters.

To calculate the monodromy data for system (3.4) we need to know the characteristic roots of $\mathcal{B}(t,\lambda)$ and their turning points. The characteristic roots $\pm\mu = \pm\mu(t,\lambda)$ are given by

$$\mu^2 = -\det \mathcal{B}(t,\lambda) = b_3^2 + (-ib_2 + b_1)(ib_2 + b_1) = b_1^2 + b_2^2 + b_3^2.$$

Using (3.8), we obtain

$$4(e^{2i\phi} - \lambda^2)^2\mu^2 = \frac{1}{4}(e^{2i\phi} - \lambda^2)(e^{2i\phi}a_\phi - \lambda^2 + 4\theta_\infty t^{-1}\lambda) + 2(\theta_1^2 - \theta_0^2)e^{i\phi}t^{-2}\lambda + 2(\theta_0^2 + \theta_1^2)e^{2i\phi}t^{-2}, \tag{3.9}$$

where $a_\phi = a_\phi(t)$ is given by

$$a_\phi = 1 - \frac{4(e^{-2i\phi}(y^*)^2 - y^2)}{y(y-1)^2} + 4e^{-i\phi}(\theta_0 + \theta_1)\frac{y+1}{y-1}t^{-1} + e^{-2i\phi}\frac{y-1}{y}((\theta_0 - \theta_1 + \theta_\infty)^2 y - (\theta_0 - \theta_1 - \theta_\infty)^2)t^{-2}. \tag{3.10}$$

By (3.9), as long as $y \neq 0, 1, \infty$ and $y^* \neq \infty$, the turning points, that is, the zeros of μ are given by the following.

Proposition 3.3. *For each t, let the square roots of $a_\phi = a_\phi(t)$ be denoted by $\pm a_\phi^{1/2}$, where $\operatorname{Re} a_\phi^{1/2} \ge 0$ and $\operatorname{Im} a_\phi^{1/2} > 0$ if $\operatorname{Re} a_\phi^{1/2} = 0$. Then, for $|\phi| < \pi/2$, the turning points are*

$$\lambda_1(t) = e^{i\phi} a_\phi^{1/2} + 2\theta_\infty t^{-1} + O(t^{-2}), \qquad \lambda_2(t) = -e^{i\phi} a_\phi^{1/2} + 2\theta_\infty t^{-1} + O(t^{-2}),$$
$$\lambda_1^0(t) = e^{i\phi} + O(t^{-2}), \qquad \lambda_2^0(t) = -e^{i\phi} + O(t^{-2})$$

as $t \to \infty$, and these are simple. Furthermore

$$\mu^2 = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_1^0)(\lambda - \lambda_2^0)}{16(\lambda - e^{i\phi})^2(\lambda + e^{i\phi})^2}.$$

Remark 3.4. For a solution $y(t)$ of $(\mathrm{P_V})$, let us consider $a_\phi(t)$ with $(y, y^*) = (y(t), y_t(t))$. On the positive real axis all the solutions corresponding to the monodromy data such that $m_{11}^0 m_{21}^0 m_{11}^1 m_{12}^1 \ne 0$ are given by [5, Theorems 3.1 and 4.1]. By using the expressions of these solutions it is easy to verify $a_0(t)$ $(= a_\phi(t)|_{\phi=0}) \ll t^{-\varepsilon}$ as $t \to \infty$ (for solutions of [5, Theorem 4.1], as $t \to \infty$ along a suitable path avoiding poles). Then $\operatorname{Re} a_\phi(t)^{1/2} \ll |t^{-\varepsilon}| + o(1)$ uniformly in t for sufficiently small $|\phi|$, which implies that, as long as $m_{11}^0 m_{21}^0 m_{11}^1 m_{12}^1 \ne 0$, every corresponding solution fulfills $0 \le \operatorname{Re} a_\phi(t)^{1/2} < 1$ if $|\phi|$ is sufficiently small. On the other hand, for a general solution in [38, Theorem 2.18] along the imaginary axis, $a_{\pi/2}(t) = 1 + O(t^{-\varepsilon})$.

Remark 3.5. To the monodromy data such that $m_{11}^0 m_{21}^0 m_{11}^1 m_{12}^1 = 0$ correspond truncated solutions in sectors containing the positive real axis [2], [4]. Let us consider $a_\phi(t)$ with $(y, y^*) = (y(t), y_t(t))$. Then we have $a_\phi(t) \ll t^{-1}$ for ϕ in some intervals containing $\phi = 0$. In the case $m_{11}^1 = 0$ the solution $y(x) \sim -1 + cx^{-1/2}e^{ix/2}$ in $0 \le \arg x \le \pi$ [4, Proposition 5], [5, Corollary 5.2] satisfies $a_\phi(t) \ll t^{-1}$ for $0 \le \phi \le \pi$. If $m_{11}^0 = m_{11}^1 = 0$, then $y(x) \sim -1 + 4(\theta_0 + \theta_1 - 1)x^{-1}$ in $|\arg x| < \pi$ [4, Proposition 2], [5, Corollary 5.3] satisfies $a_\phi(t) \ll t^{-2}$ for $|\phi| < \pi$. For $m_{12}^1 = 0$ or $m_{21}^0 = 0$, truncated solutions such that $a_\phi(t) \ll t^{-1}$ for $|\phi| < \pi/2$ are given by [2].

By (3.9) the characteristic root $\mu = \mu(t, \lambda)$ is written in the form

$$\mu = \frac{1}{4}\sqrt{\frac{e^{2i\phi}a_\phi - \lambda^2}{e^{2i\phi} - \lambda^2}} + \frac{\theta_\infty \lambda t^{-1}}{2\sqrt{(e^{2i\phi} - \lambda^2)(e^{2i\phi}a_\phi - \lambda^2)}} + g_2(t, \lambda)t^{-2} \tag{3.11}$$

as $t \to \infty$. Here $g_2(t, \lambda)$ has branch points at $\lambda_{1,2}^0$, $\lambda_{1,2}$, $\pm e^{i\phi}$ and $\pm e^{i\phi}a_\phi^{1/2}$, but it fulfills $g_2(t, \lambda) \ll 1$ if $|\lambda^2 - e^{2i\phi}a_\phi|^{-1} + |\lambda^2 - e^{2i\phi}|^{-1} \ll 1$. The algebraic function $\mu(t, \lambda)$ is given on the Riemann surface consisting of two copies of λ-plane $\mathbb{P}_+$ and $\mathbb{P}_-$ glued along the cuts $[\lambda_1, \lambda_1^0]$, $[\lambda_2^0, \lambda_2]$ (cf. Figure 3.2, (a)). In (3.11), each square root is fixed in such a way that

$$\sqrt{\frac{e^{2i\phi}a_\phi - \lambda^2}{e^{2i\phi} - \lambda^2}} \to 1, \quad \lambda^{-2}\sqrt{(e^{2i\phi}a_\phi - \lambda^2)(e^{2i\phi} - \lambda^2)} \to -1$$

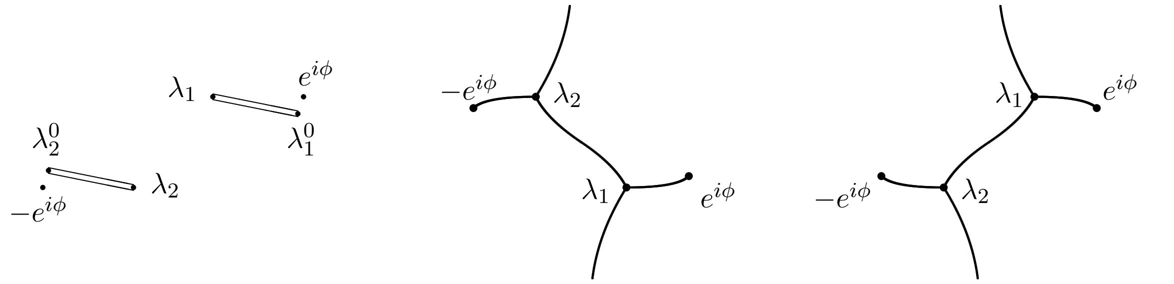
as $\lambda \to \infty$ on $\mathbb{P}_+$. Then, for $a_\phi^{1/2}$ as in Proposition 3.3, $a_\phi^{-1/2}\sqrt{(e^{2i\phi}a_\phi - \lambda^2)/(e^{2i\phi} - \lambda^2)}$, $e^{-2i\phi}a_\phi^{-1/2}\sqrt{(e^{2i\phi}a_\phi - \lambda^2)(e^{2i\phi} - \lambda^2)} \to 1$ as $\lambda \to 0$ on $\mathbb{P}_+$.

A Stokes curve is defined by

$$\mathrm{Re} \int_{\lambda_*}^{\lambda} \mu(t,\tau)d\tau = 0,$$

where λ_* is a turning point [11]. This curve connects λ_* to another turning point, $\pm e^{i\phi}$ or ∞. The Stokes graph consists of Stokes curves, turning points and singular points.

Our WKB analysis will be carried out under the supposition $a_\phi(t) \to A_\phi$ as $t \to \infty$ (cf. (4.3)), where A_ϕ is a unique solution of the Boutroux equations (2.3). Suppose that $0 < |\phi| < \pi/2$. Let us consider the *limit* Stokes graph for $t = \infty$, in which the limit turning points $\lambda_{1,2}(\infty)$ are also denoted by the same symbols $\lambda_{1,2}$ as for $t \neq \infty$. Set $\mathbb{P}_+^\infty \cup \mathbb{P}_-^\infty = \lim_{t\to\infty} \mathbb{P}_+ \cup \mathbb{P}_-$, which is a two-sheeted Riemann surface glued along the cuts $[-e^{i\phi}, \lambda_2]$, $[\lambda_1, e^{i\phi}]$ with $\lambda_{1,2} = \lambda_{1,2}(\infty) = \pm e^{i\phi} A_\phi^{1/2}$ and $\lambda^0_{1,2}(\infty) = \pm e^{i\phi}$. As long as the turning points do not coalesce, the limit Stokes curve for $t = \infty$ reflects the Boutroux equations. Then this limit Stokes graph on $\mathbb{P}_+^\infty$ is considered to be as in Figure 3.2 (b), (c) (cf. Proposition 8.17), in which, if $0 < \phi < \pi/2$, the limit Stokes curves connect λ_1 to $e^{i\phi}$, λ_2 and $i\infty$, and λ_2 to $-e^{i\phi}$ λ_1 and $-i\infty$.



(a) $[\lambda_1, \lambda_1^0]$, $[\lambda_2^0, \lambda_2]$ (b) $-\pi/2 < \phi < 0$ (c) $0 < \phi < \pi/2$

FIGURE 3.2. Cuts on $\mathbb{P}_+$ and the limit Stokes graph on $\mathbb{P}_+^\infty$

An unbounded domain $\mathcal{D} \subset \mathbb{P}_+^\infty \cup \mathbb{P}_-^\infty$ is called a canonical domain if, for each $\lambda \in \mathcal{D}$, there exist contours $\mathcal{C}_\pm(\lambda) \subset \mathcal{D}$ ending at λ such that

$$\mathrm{Re} \int_{\lambda_-}^{\lambda} \mu(\tau)d\tau \to -\infty \quad \left(\text{respectively, } \mathrm{Re} \int_{\lambda_+}^{\lambda} \mu(\tau)d\tau \to +\infty\right) \tag{3.12}$$

as $\lambda_- \to \infty$ along $\mathcal{C}_-(\lambda)$ (respectively, as $\lambda_+ \to \infty$ along $\mathcal{C}_+(\lambda)$) (see [11], [12, p. 242]). The interior of a canonical domain contains exactly one Stokes curve, and its boundary consists of Stokes curves.

3.4. **WKB-solution.** The following is a WKB-solution of (3.4) in a canonical domain.

Proposition 3.4. *In the canonical domain whose interior contains a Stokes curve issuing from the turning point λ_1 or λ_2, system* (3.4) *with $\hat{u} \equiv 1$ admits a solution expressed by*

$$\Psi_{\mathrm{WKB}}(\lambda) = T(I + O(t^{-\delta})) \exp\left(\int_{\tilde{\lambda}_*}^{\lambda} \Lambda(\tau)d\tau\right)$$

outside suitable neighbourhoods of zeros and poles of $b_1 \pm ib_2$ *as long as* $|\lambda \pm e^{i\phi}| \gg t^{-2+2\delta}$, $|\lambda - \lambda_\iota| \gg t^{-2/3+(2/3)\delta}$ $(\iota = 1, 2)$, $0 < \delta < 1$ *being arbitrary. Here* $\tilde{\lambda}_*$ *is a base point near* λ_1 *or* λ_2*, and* $\Lambda(\tau)$ *and* T *are given by*

$$\Lambda(\lambda) = t\mu\sigma_3 - \mathrm{diag}T^{-1}T_\lambda, \quad T = \begin{pmatrix} 1 & \dfrac{b_3 - \mu}{b_1 + ib_2} \\ \dfrac{\mu - b_3}{b_1 - ib_2} & 1 \end{pmatrix}.$$

Remark 3.6. In this proposition

$$\begin{aligned}\mathrm{diag}T^{-1}T_\lambda &= \frac{1}{2\mu(\mu + b_3)}\big(i(b_1b_2' - b_1'b_2)\sigma_3 + (b_3\mu' - b_3'\mu)I\big) \qquad (b_\iota' = \partial b_\iota/\partial\lambda) \\ &= \frac{1}{4}\Big(1 - \frac{b_3}{\mu}\Big)\frac{\partial}{\partial\lambda}\log\frac{b_1 + ib_2}{b_1 - ib_2}\sigma_3 + \frac{1}{2}\frac{\partial}{\partial\lambda}\log\frac{\mu}{\mu + b_3}I.\end{aligned}$$

Proof. By $Y = T\tilde{Y}$ system (3.4) with $\hat{u} \equiv 1$ becomes

$$\tilde{Y}_\lambda = (t\mu\sigma_3 - T^{-1}T_\lambda)\tilde{Y}. \tag{3.13}$$

To remove the off-diagonal part $R = T^{-1}T_\lambda - \mathrm{diag}T^{-1}T_\lambda$ set

$$T_1 = \frac{1}{2t\mu}\begin{pmatrix} 0 & R_{12} \\ -R_{21} & 0 \end{pmatrix}, \quad R_{12} = \frac{\mu + b_3}{2\mu}\frac{\partial}{\partial\lambda}\Big(\frac{b_3 - \mu}{b_1 + ib_2}\Big), \quad R_{21} = \frac{\mu + b_3}{2\mu}\frac{\partial}{\partial\lambda}\Big(\frac{\mu - b_3}{b_1 - ib_2}\Big),$$

which fulfills $[t\mu\sigma_3, T_1] = R$. Now we would like to find X such that the transformation $\tilde{Y} = (I + T_1)(I + X)Z$ takes (3.13) to

$$Z_\lambda = \Lambda Z = (t\mu\sigma_3 - \mathrm{diag}T^{-1}T_\lambda)Z,$$

that is,

$$(T_1)_\lambda(I + X) + (I + T_1)X_\lambda + (I + T_1)(I + X)\Lambda = (\Lambda - R)(I + T_1)(I + X).$$

It follows that

$$X_\lambda = [\Lambda, X] + (I + T_1)^{-1}Q(I + X) \tag{3.14}$$

with $Q = -(T_1)_\lambda - R(I + T_1) + [\Lambda, T_1] = -(T_1)_\lambda - T^{-1}T_\lambda T_1 + T_1\mathrm{diag}T^{-1}T_\lambda$. Then $\|Q\|$ is estimated as follows:

(1) Near $\lambda = \mp e^{i\phi}$, we have b_3, $|b_1 \pm ib_2| \ll |\lambda \pm e^{i\phi}|^{-1}$, $|b_1b_2' - b_1'b_2| \ll |\lambda \pm e^{i\phi}|^{-2}$, $\mu \asymp |\lambda \pm e^{i\phi}|^{-1/2}$, and hence $\|R\| \ll |\lambda \pm e^{i\phi}|^{-1} \asymp \mu^2$, $\|\mathrm{diag}T^{-1}T_\lambda\| \ll |\lambda \pm e^{i\phi}|^{-1}$, $\|T_1\| \ll t^{-1}|\lambda \pm e^{i\phi}|^{-1/2}$ and $\|Q\| \ll t^{-1}|\lambda \pm e^{i\phi}|^{-3/2}$.

(2) Near $\lambda = \lambda_\iota$ $(\iota = 1, 2)$ we have b_3, $|b_1b_2' - b_1'b_2| \ll 1$, $\mu \asymp |\lambda - \lambda_\iota|^{1/2}$, and hence $\|R\| \ll |\lambda - \lambda_\iota|^{-1}$, $\|\mathrm{diag}T^{-1}T_\lambda\| \ll |\lambda - \lambda_\iota|^{-1}$, $\|T_1\| \ll t^{-1}|\lambda - \lambda_\iota|^{-3/2}$, and $\|Q\| \ll t^{-1}|\lambda - \lambda_\iota|^{-5/2}$.

(3) Near $\lambda = \infty$, observe that $\mu = 1/4 + O(\lambda^{-1})$ on $\mathbb{P}_+^\infty$ and $b_3 = 1/4 + O(\lambda^{-1})$. Then $\pm(b_3 - \mu)/(b_1 \pm ib_2) = -(b_1 \mp ib_2)/(\mu + b_3) \ll \lambda^{-1}$, which means $T = I + O(\lambda^{-1})$. It is easy to see that $\|T_1\| \ll t^{-1}\lambda^{-2}$, $\|Q\| \ll t^{-1}\lambda^{-3}$ near $\lambda = \infty$ on $\mathbb{P}_+^\infty$. Near ∞ on $\mathbb{P}_-^\infty$, $\|T_1\| \ll t^{-1}\lambda^{-1}$, $\|Q\| \ll t^{-1}\lambda^{-2}$, since $\mu + 1/4 \ll t^{-1}\lambda^{-1}$.

The σ_3-component of $\mathrm{diag}T^{-1}T_\lambda$ is $D_3(\lambda)=\frac{1}{4}(1-b_3/\mu)(\log((b_1+ib_2)/(b_1-ib_2)))_\lambda\sigma_3$ (cf. Remark 3.6), which satisfies $\|D_3(\lambda)\|\ll|\lambda\mp e^{i\phi}|^{-1/2}$ near $\lambda=\pm e^{i\phi}$; $\ll|\lambda-\lambda_\iota|^{-1/2}$ near $\lambda=\lambda_\iota$; and $\ll\lambda^{-2}$ near $\lambda=\infty$.

Every solution of the integral equation

$$\begin{aligned}X(\lambda)&=\int_{\mathcal{C}(\lambda)}\exp\Big(\int_\xi^\lambda\Lambda(\tau)d\tau\Big)(I+T_1(\xi))^{-1}Q(\xi)(I+X(\xi))\exp\Big(-\int_\xi^\lambda\Lambda(\tau)d\tau\Big)d\xi\\&=\int_{\mathcal{C}(\lambda)}e^{(tM(\lambda,\xi)-J(\lambda,\xi))\sigma_3}(I+T_1(\xi))^{-1}Q(\xi)(I+X(\xi))e^{-(tM(\lambda,\xi)-J(\lambda,\xi))\sigma_3}d\xi\end{aligned}$$

with

$$M(\lambda,\xi)=\int_\xi^\lambda\mu(\tau)d\tau,\quad J(\lambda,\xi)=\int_\xi^\lambda D_3(\tau)d\tau$$

solves (3.14), where the set of contours $\mathcal{C}(\lambda)$ ending in λ is chosen in such a way that for (1,2)- (respectively, (2,1)-) entry is $\mathcal{C}_-(\lambda)$ (respectively, $\mathcal{C}_+(\lambda)$) (cf. (3.12)), and that those for (1,1)- and (2,2)-entries are paths joining λ_ι to λ. In the canonical domain $J(\lambda,\xi)$ is bounded uniformly as long as λ and ξ are outside neighbourhoods of $\pm e^{i\phi}$ and λ_ι. In the canonical domain, as long as $|\lambda\pm e^{i\phi}|\gg t^{-2(1-\delta)}$, $|\lambda-\lambda_\iota|\gg t^{-(2/3)(1-\delta)}$, we have $\|T_1\|\ll t^{-\delta}$, and

$$\|X\|\ll\|Q\|\ll t^{-1}(|\lambda\pm e^{i\phi}|^{-1/2}+|\lambda-\lambda_1|^{-3/2}+|\lambda-\lambda_2|^{-3/2}+1)\ll t^{-\delta},$$

which implies the proposition (cf. the proof of [12, Theorem 7.2]). □

3.5. **Local solutions around turning points.** For $\iota=1$ or 2, if $|\lambda-\lambda_\iota|\ll t^{-2/3}$, the WKB-solution given above fails in expressing the asymptotic behaviour. Consider the system

$$\frac{dW}{d\zeta}=\begin{pmatrix}0&1\\ \zeta&0\end{pmatrix}W,\tag{3.15}$$

which admits canonical matrix solutions $W_\nu(\zeta)$ ($\nu=0,\pm1,\pm2,\ldots$) such that

$$W_\nu(\zeta)=\zeta^{-(1/4)\sigma_3}(\sigma_3+\sigma_1)(I+O(\zeta^{-3/2}))\exp\big(\tfrac{2}{3}\zeta^{3/2}\sigma_3\big)$$

as $\zeta\to\infty$ through the sector $\Sigma_\nu:|\arg\zeta-(2\nu-1)\pi/3|<2\pi/3$, and that $W_{\nu+1}(\zeta)=W_\nu(\zeta)G_\nu$ with

$$G_1=\begin{pmatrix}1&-i\\0&1\end{pmatrix},\quad G_2=\begin{pmatrix}1&0\\-i&1\end{pmatrix},\quad G_{\nu+1}=\sigma_1G_\nu\sigma_1.$$

In particular

$$W_1(\zeta)=\begin{pmatrix}\mathrm{Bi}(\zeta)&\mathrm{Ai}(\zeta)\\ \mathrm{Bi}_\zeta(\zeta)&\mathrm{Ai}_\zeta(\zeta)\end{pmatrix},$$

where $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ are the Airy functions [1, 10] such that $\mathrm{Ai}(\zeta)\sim\zeta^{-1/4}\exp(-\frac{2}{3}\zeta^{3/2})$ as $\zeta\to\infty$ in $|\arg\zeta|<\pi$ and that $\mathrm{Bi}(\zeta)=\omega^{-1/4}\mathrm{Ai}(\omega^{-1}\zeta)$ with $\omega=e^{2\pi i/3}$. Then we have the following local solution around each turning point.

Proposition 3.5. *For each turning point* λ_ι $(\iota = 1, 2)$ *write* $c_k = b_k(\lambda_\iota)$, $c'_k = (b_k)_\lambda(\lambda_\iota)$ $(k = 1, 2, 3)$, *and suppose that* c_k, c'_k *are bounded and* $c_1 \pm ic_2 \neq 0$. *Let* $W(\zeta)$ *be a given matrix solution of* (3.15), *and let* $\lambda - \lambda_\iota = (2\kappa)^{-1/3} t^{-2/3}(\zeta + \zeta_0)$ *with* $\kappa = c_1c'_1 + c_2c'_2 + c_3c'_3$, $|\zeta_0| \ll t^{-1/3}$. *Then system* (3.4) *with* $\hat{u} \equiv 1$ *admits a matrix solution given by*

$$\Phi_\iota(\lambda) = T_\iota(I + O(t^{-\delta'})) \begin{pmatrix} 1 & 0 \\ 0 & \hat{t}^{-1} \end{pmatrix} W(\zeta), \quad T_\iota = \begin{pmatrix} 1 & -\dfrac{c_3}{c_1 + ic_2} \\ -\dfrac{c_3}{c_1 - ic_2} & 1 \end{pmatrix}$$

with $\hat{t} = 2(2\kappa)^{-1/3}(c_1 - ic_2)t^{1/3}$ *as long as* $|\zeta| \ll t^{1/3-\delta'/3}$, *that is,* $|\lambda - \lambda_\iota| \ll t^{-1/3-\delta'/3}$, $0 < \delta' < 1$ *being arbitrary.*

Proof. Since $\mu^2 = b_1^2 + b_2^2 + b_3^2$, we have $c_1^2 + c_2^2 + c_3^2 = \mu(\lambda_\iota)^2 = 0$. Write $\mathcal{B}(t, \lambda) = \mathcal{B}_0(t) + \mathcal{B}_1(t, \lambda)$ with

$$\mathcal{B}_0(t) = \mathcal{B}(t, \lambda_\iota) = \begin{pmatrix} c_3 & c_1 - ic_2 \\ c_1 + ic_2 & -c_3 \end{pmatrix}, \quad \mathcal{B}_1(t, \lambda) = \begin{pmatrix} \delta_3 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & -\delta_3 \end{pmatrix},$$

where $\delta_k = b_k - c_k$ $(k = 1, 2, 3)$. Set $\eta = \lambda - \lambda_\iota$. Observing that $\delta_k = \sum_{j \geq 1} c_{k,j}\eta^j$ with $c_{k,1} = c'_k$, and that

$$T_\iota^{-1}\mathcal{B}_0(t)T_\iota = \begin{pmatrix} 0 & 2(c_1 - ic_2) \\ 0 & 0 \end{pmatrix},$$

$$T_\iota^{-1}\mathcal{B}_1(t, \lambda)T_\iota = \begin{pmatrix} ic_3^{-1}(c_2\delta_1 - c_1\delta_2) & (c_1 + ic_2)^{-1}(c_1\delta_1 + c_2\delta_2 - c_3\delta_3) \\ (c_1 - ic_2)^{-1}(c_1\delta_1 + c_2\delta_2 + c_3\delta_3) & -ic_3^{-1}(c_2\delta_1 - c_1\delta_2) \end{pmatrix},$$

we have

$$T_\iota^{-1}\mathcal{B}(t, \lambda)T_\iota = \begin{pmatrix} 0 & \gamma_0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f(\eta) & g(\eta) \\ h(\eta) & -f(\eta) \end{pmatrix} \eta,$$

where $f(\eta)$, $g(\eta)$ and $h(\eta)$ are analytic around $\eta = 0$ and

$$\gamma_0 = 2(c_1 - ic_2), \quad f(0) = ic_3^{-1}(c'_1c_2 - c_1c'_2),$$
$$g(0) = (c_1 + ic_2)^{-1}(c_1c'_1 + c_2c'_2 - c_3c'_3), \quad h(0) = (c_1 - ic_2)^{-1}(c_1c'_1 + c_2c'_2 + c_3c'_3).$$

Note that

$$\begin{pmatrix} f(\eta)\eta & \gamma_0 + g(\eta)\eta \\ h(\eta)\eta & -f(\eta)\eta \end{pmatrix} (I + H(\eta)) = (I + H(\eta)) \begin{pmatrix} 0 & \gamma_0 \\ h_*(\eta)\eta & 0 \end{pmatrix},$$

in which $H(\eta) = L(\eta)\eta$ with

$$L(\eta) = \begin{pmatrix} l(\eta) & 0 \\ l_*(\eta) & -l(\eta) \end{pmatrix} := \frac{1}{2\gamma_0 + g(\eta)\eta} \left(\begin{pmatrix} 0 & 0 \\ -2f(\eta) & 0 \end{pmatrix} + g(\eta)\sigma_3 \right),$$

$$h_*(\eta) = \frac{h(\eta)(1 + l(\eta)\eta) - f(\eta)l_*(\eta)\eta}{1 - l(\eta)\eta} = h(\eta) + O(\eta).$$

By

$$Y = T_\iota(I + H(\eta))Z = T_\iota(I + L(\eta)\eta)Z, \quad L(0) = \begin{pmatrix} q & 0 \\ p & -q \end{pmatrix}$$

system (3.4) is changed into

$$\frac{dZ}{d\eta} = t\mathcal{B}_\iota(t, \eta)Z,$$

$$\mathcal{B}_\iota(t, \eta) = \begin{pmatrix} 0 & \delta_0 \\ h_*(\eta)\eta & 0 \end{pmatrix} - t^{-1}(I + L(\eta)\eta)^{-1}(L(\eta)\eta)'$$

$$= \begin{pmatrix} 0 & 2(c_1 - ic_2) \\ \kappa(c_1 - ic_2)^{-1}\eta h_0(\eta) & 0 \end{pmatrix} + \begin{pmatrix} -q & 0 \\ -p & q \end{pmatrix} t^{-1} + t^{-1}\sum_{j\ge 1} B_j^*\eta^j,$$

where $h_0(\eta) = h_*(\eta)h(0)^{-1} = 1 + \eta h_1(\eta) = 1 + O(\eta)$. Let $\varphi(\xi)$ be such that $\varphi(0) = 1$ and that $\varphi(\xi)h_0(\xi\varphi(\xi))((\xi\varphi(\xi))')^2 = \varphi(\xi)h_0(\xi\varphi(\xi))(\varphi(\xi) + \xi\varphi'(\xi))^2 = 1$. Then the change of variables $\eta = \xi\varphi(\xi)$ takes this system to

$$\frac{dZ}{d\xi} = t\Big(\hat{\mathcal{B}}_{\iota,0}(t, \xi) + (1 + O(\xi))t^{-1}\Big(L(0) + \sum_{j\ge 1} \hat{B}_j^*\xi^j\Big)\Big)Z$$

with

$$\hat{\mathcal{B}}_{\iota,0}(t, \xi) = \begin{pmatrix} 0 & 2(c_1 - ic_2)(\xi\varphi(\xi))' \\ \kappa(c_1 - ic_2)^{-1}\xi\varphi(\xi)h_0(\xi\varphi(\xi))(\xi\varphi(\xi))' & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2(c_1 - ic_2)(\xi\varphi(\xi))' \\ \kappa(c_1 - ic_2)^{-1}\xi/(\xi\varphi(\xi))' & 0 \end{pmatrix}.$$

Apply the additional transformation $Z = \mathrm{diag}[1, 1/(\xi\varphi(\xi))']V$, and set $\xi = \eta + O(\eta^2) = \beta z$ with $\beta = (2\kappa)^{-1/3}t^{-2/3}$ and $\hat{t} = 2(c_1 - ic_2)\beta t \asymp t^{1/3}$. Then

$$\frac{dZ}{dz} = \left(\begin{pmatrix} -q\beta & \hat{t} \\ \hat{t}^{-1}z - p\beta & q\beta \end{pmatrix} + O(\hat{t}^{-4}z)\right)Z \tag{3.16}$$

as long as $|\beta z| \asymp t^{-2/3}z$ is sufficiently small. The further change of variables

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & \hat{t}^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q\beta & 1 \end{pmatrix}V, \quad \zeta = z - \zeta_0, \quad \zeta_0 = p\beta\hat{t} - q^2\beta^2 \ll t^{-1/3}$$

yields

$$\frac{dV}{d\zeta} = \left(\begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} + \Delta(t, \zeta)\right)V, \quad \Delta(t, \zeta) \ll t^{-1}(|\zeta| + |t^{-1/3}|). \tag{3.17}$$

Suppose that $V = P(t, \zeta)W$ reduces (3.17) to (3.15). Then $P = P(t, \zeta)$ satisfies

$$\frac{dP}{d\zeta} = \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix}P - P\begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} + \Delta(t, \zeta)P. \tag{3.18}$$

Note that, in each sector Σ_ν $(\nu = 0, \pm 1)$, the function $P_\nu(\zeta) = I + X_\nu(\zeta)$ such that

$$X_\nu(\zeta) = \int_{\mathcal{C}_\nu(\zeta)} W_\nu(\zeta) W_\nu(\xi)^{-1} \Delta(t,\xi)(I + X_\nu(\xi)) W_\nu(\xi) W_\nu(\zeta)^{-1} d\xi \tag{3.19}$$

solves (3.18) in Σ_ν. Here $\mathcal{C}_\nu(\zeta)$ is a set of contours $\gamma(\zeta)$ ending at ζ for each term of the integrand chosen according to the multiplier $g(\xi,\zeta)$ caused by $W_\nu(\zeta)W_\nu(\xi)^{-1}$, which belongs to $G_0 \cup G_+ \cup G_-$ up to $(1 + O(|\zeta|^{-3/2} + |\xi|^{-3/2}))$, where

$$G_0 = \{1,\ \xi^{\pm 1/2},\ \zeta^{\pm 1/2},\ (\xi\zeta)^{\pm 1/2},\ (\xi/\zeta)^{\pm 1/2}\}, \quad G_\pm = \{\rho \exp(\pm \tfrac{4}{3}(\zeta^{3/2} - \xi^{3/2})) \mid \rho \in G_0\}.$$

The choice of $\gamma(\zeta)$ is described as follows: (i) for $g(\xi,\zeta) \in G_0$ let $\gamma(\zeta)$ be a segment joining 0 to ζ; and (ii) if, say, $\zeta \in \Sigma_1$, for $g(\xi,\zeta) \in G_+$ (respectively, $\in G_-$) choose $\gamma(\zeta)$ to be a line issuing from ζ and tending to $+\infty$ (respectively, $-\infty$). Then it is easy to see that $\int_{\gamma(\zeta)} g(\xi,\zeta) d\xi \ll \zeta^2$. For $\nu = 0$, ± 1, from (3.19) we derive $\|X_\nu(\zeta)\| \ll \zeta^2 \Delta(t,\zeta) \ll |t^{-1}\zeta^3| + |t^{-4/3}\zeta^2| \ll t^{-\delta'}$ as long as $|\zeta| \ll t^{1/3-\delta'/3}$ with $0 < \delta' < 1$, and obtain the solutions $P_\nu(\zeta) = I + X_\nu(\zeta) = I + O(t^{-\delta'})$ of (3.18) in $\Sigma_\nu \cap \{|\zeta| \ll t^{1/3-\delta'/3}\}$. It remains to show that there exists $Q(t,\zeta) = I + O(t^{-\delta'})$ in $|\zeta| \ll t^{1/3-\delta'/3}$ such that $V = Q(t,\zeta)W$ reduces (3.17) to (3.15) by using $P_\nu(\zeta)$'s. This uniform reduction is constructed by a parallel argument as in [42, Section 6.5] in which $\epsilon^{-2/3}x$, $\mathrm{diag}[1, \epsilon^{-1/3}]z$, Σ_j and " ~ 1 " correspond to our ζ, W, Σ_{j-1} and " $= I + O(t^{-\delta'})$ ", respectively. Thus, for given $W(\zeta)$, we obtain a solution as in the proposition. □

Remark 3.7. (1) Let $E(t) := \{\zeta \mid |\zeta| \ll t^{1/3-\delta'/3},\ |\exp(\pm\tfrac{2}{3}\zeta^{3/2})| \ll 1\}$. For every $g(\xi,\zeta) \in G_0 \cup G_+ \cup G_-$, we also have $\int_0^\zeta g(\xi,\zeta) d\zeta \ll \zeta^2$ uniformly in $E(t)$. Thus we may easily derive a solution of Proposition 3.5 in $E(t)$ without using the reasoning of [42]. Since $E(t)$ contains all Stokes curves $\zeta = re^{\frac{1}{3}(2k-1)i}$, $k = 0, \pm 1$, $r > 0$, this solution restricted to $E(t)$ is enough for our use, in which matchings are along Stokes curves.

(2) In the proof above, without the change of variables $\eta = \xi\varphi(\xi)$, we have $\Delta(t,\zeta) \ll (|t^{-2/3}\zeta^2| + |t^{-1}\zeta| + |t^{-4/3}|)$ caused by the η^2 term of $h_*(\eta)$, and the estimates in the proposition are $|\zeta| \ll t^{1/6-\delta'/6}$ and $|\lambda - \lambda_\iota| \ll t^{-1/2-\delta'/6}$.

4. Monodromy matrices

We would like to find the monodromy matrices M^0, M^1 with respect to the matrix solution (3.6). Let M^0_* and M^1_* be the monodromy matrices in the case where (3.6) solves (3.4) with $\hat{u} \equiv 1$. Since the gauge transformation $Y = \hat{u}^{\sigma_3/2} Y_*$ reduces (3.4) to the system with $\hat{u} \equiv 1$, the monodromy matrices M^0, M^1 in the case of general $\hat{u}$ are given by

$$M^0 = \hat{u}^{\sigma_3/2} M^0_* \hat{u}^{-\sigma_3/2}, \quad M^1 = \hat{u}^{\sigma_3/2} M^1_* \hat{u}^{-\sigma_3/2}.$$

In this section we calculate M^0_* and M^1_* by using connection matrices and S^*_1, S^*_2 such that

$$S_1 = \hat{u}^{\sigma_3/2} S^*_1 \hat{u}^{-\sigma_3/2}, \quad S_2 = \hat{u}^{\sigma_3/2} S^*_2 \hat{u}^{-\sigma_3/2}.$$

4.1. **Stokes graph.** Recall the Riemann surface $\mathbb{P}^\infty_+ \cup \mathbb{P}^\infty_-$ glued along the cuts $[-e^{i\phi}, \lambda_2]$ and $[\lambda_1, e^{i\phi}]$ with $\lambda_{2,1} = \mp e^{i\phi} A_\phi^{1/2}$ for $t = \infty$. As in Figure 4.1 (a), (b), symbols are assigned to the Stokes curves of the limit Stokes graph on $\mathbb{P}^\infty_+$. For $-\pi/2 < \phi < 0$, let $\mathbf{c}_2^\infty$, $\hat{\mathbf{c}}_1^\infty$, $\mathbf{c}_0$ be Stokes curves on $\mathbb{P}^\infty_+$ connecting λ_2 to $i\infty$, λ_1 to $-i\infty$, λ_2 to λ_1, respectively, and, for $0 < \phi < \pi/2$, $\mathbf{c}_1^\infty$, $\hat{\mathbf{c}}_2^\infty$, $\mathbf{c}_0$ connecting λ_1 to $i\infty$, λ_2 to $-i\infty$, λ_2 to λ_1, respectively.

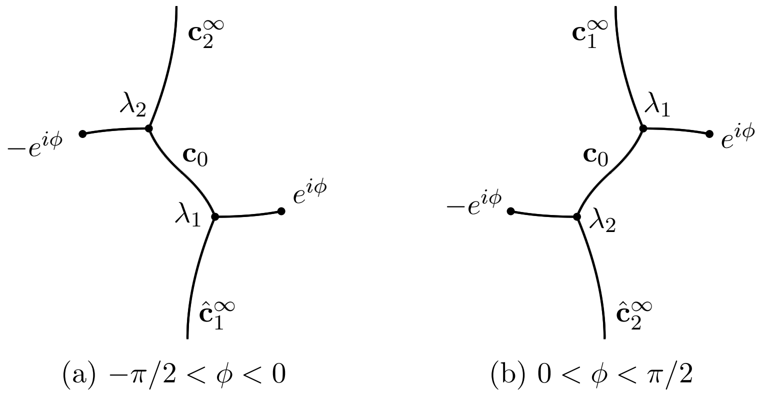


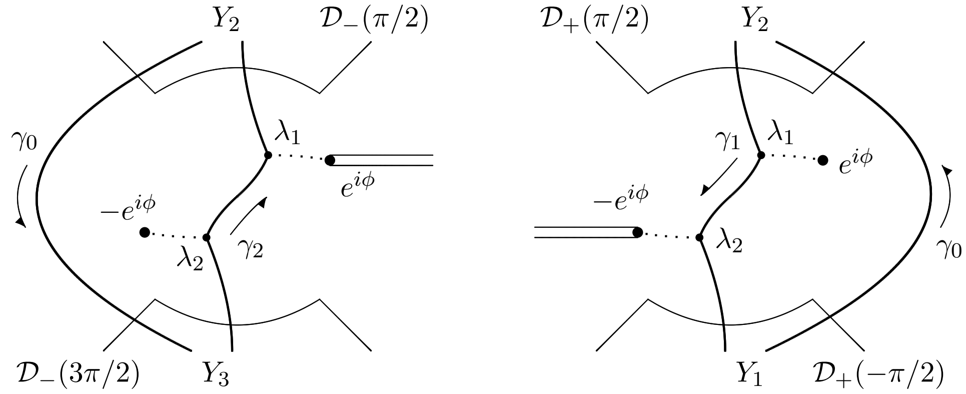
FIGURE 4.1. Limit Stokes graph

4.2. **Connection matrices.** Recall the canonical solutions $Y(t,\lambda) = Y_2(t,\lambda)$ and $Y_3(t,\lambda)$ in the sectors $|\arg\lambda - \pi/2| < \pi$ and $|\arg\lambda - 3\pi/2| < \pi$, respectively, near $\lambda = \infty$. For (3.4) with $\hat{u} \equiv 1$, the Stokes matrix S_2^* is given by $Y_3 = YS_2^*$ (cf. (2.2)). Set $\mathcal{D}_-^0 = \mathbb{C} \setminus [e^{i\phi}, +\infty]$ with $0 < \arg\lambda < 2\pi$. Let us consider the case, say, $0 < \phi < \pi/2$. For the Stokes curve $\gamma_2 := (-\hat{\mathbf{c}}_2^\infty) \cup \mathbf{c}_0 \cup \mathbf{c}_1^\infty$ on $\mathbb{P}^\infty_+$, the projection $\mathrm{pr}(\gamma_2)$ to $\mathcal{D}_-^0$ is a path issuing from $e^{3\pi i/2}\infty$ and ending in $e^{\pi i/2}\infty$ as in Figure 4.2 (a). In what follows let us identify $\mathrm{pr}(\gamma_2)$ with γ_2 and denote by the same symbol. Note that $Y = Y_2$ (respectively, Y_3) is analytic in the domain $\mathcal{D}_-(\pi/2) := \{|\arg\lambda - \pi/2| < \pi/4,\ |\lambda| > 2025\} \subset \mathcal{D}_-^0$ (respectively, $\mathcal{D}_-(3\pi/2) := \{|\arg\lambda - 3\pi/2| < \pi/4,\ |\lambda| > 2025\} \subset \mathcal{D}_-^0$), which is simply connected. The loop $\hat{l}_0$ of Figure 3.1 with $\hat{p}_{\mathrm{st}} \in \mathcal{D}_-(\pi/2)$ is decomposed into $\hat{l}_0 = \gamma_2 \circ \gamma_0$, where γ_0 is an arc lying in $\{|\lambda| > 2025\} \cap \mathcal{D}_-^0$ issuing from $\mathcal{D}_-(\pi/2)$ and ending in $\mathcal{D}_-(3\pi/2)$ as in Figure 4.2 (a). The analytic continuation of Y_3 along γ_2 results in $Y_3^{\gamma_2}\Gamma^\infty_{\infty\,2} = Y$ in $\mathcal{D}_-(\pi/2)$, where $\Gamma^\infty_{\infty\,2} \in SL_2(\mathbb{C})$ is a connection matrix. On the other hand the relation $YS_2^* = Y_3$ is also written in the form $Y^{\gamma_0}S_2^* = Y_3$ in $\mathcal{D}_-(3\pi/2)$, where Y^{γ_0} denotes the analytic continuation of Y along γ_0. Then the analytic continuation of both sides along γ_2 yields $Y^{\hat{l}_0}S_2^* = Y^{\gamma_2\circ\gamma_0}S_2^* = Y_3^{\gamma_2} = Y(\Gamma^\infty_{\infty\,2})^{-1}$, i.e., $Y^{\hat{l}_0} = Y(\Gamma^\infty_{\infty\,2})^{-1}(S_2^*)^{-1} = YM_*^0$ in $\mathcal{D}_-(\pi/2)$, which implies

$$\Gamma^\infty_{\infty\,2} M_*^0 = (S_2^*)^{-1}. \tag{4.1}$$

In the domain $\mathcal{D}_+^0 = \mathbb{C} \setminus [-\infty, -e^{i\phi}]$ with $|\arg\lambda| < \pi$, for the Stokes curve $\gamma_1 = (-\mathbf{c}_1^\infty) \cup (-\mathbf{c}_0) \cup \hat{\mathbf{c}}_2^\infty$ on $\mathbb{P}^\infty_+$, considering the projection to $\mathcal{D}_+^0$ issuing from $e^{\pi i/2}\infty$ and ending in $e^{-\pi i/2}\infty$, we analogously obtain

$$M_*^1 \Gamma^\infty_{\infty\,1} = (S_1^*)^{-1}, \tag{4.2}$$



(a) $\mathcal{D}_-^0 = \mathbb{C} \setminus [e^{i\phi}, +\infty]$ (b) $\mathcal{D}_+^0 = \mathbb{C} \setminus [-\infty, -e^{i\phi}]$

FIGURE 4.2. Composite loops in the case $0 < \phi < \pi/2$

where $\Gamma^{\infty}_{\infty 1}$ is a connection matrix such that $Y^{\gamma_1}\Gamma^{\infty}_{\infty 1} = Y_1$ in $\mathcal{D}_+(-\pi/2) := \{|\arg\lambda + \pi/2| < \pi/4,\ |\lambda| > 2025\} \subset \mathcal{D}_+^0$. Our concern is finding the connection matrices $\Gamma^{\infty}_{\infty 1}$ and $\Gamma^{\infty}_{\infty 2}$.

Let us calculate $\Gamma^{\infty}_{\infty 2}$ in the case $0 < \phi < \pi/2$. To discuss according to the justification scheme of Kitaev [26], suppose that $a_\phi = a_\phi(t)$ is given by (3.10) with $(y, y^*) = (y(t), y^*(t))$ not necessarily solving $(\mathrm{P_V})$, and that

$$a_\phi(t) = A_\phi + t^{-1}B_\phi(t), \quad B_\phi(t) = O(1) \tag{4.3}$$

for $t \in S_*(\phi, t'_\infty, \kappa_0, \delta_1)$ with given κ_0, given small δ_1 and sufficiently large t'_∞. Here A_ϕ is a unique solution of the Boutroux equations (2.3), and

$$S_*(\phi, t'_\infty, \kappa_0, \delta_1) = \{t \mid \mathrm{Re}\, t > t'_\infty, |\mathrm{Im}\, t| < \kappa_0, |y^*(t)| + |y(t)| + |y(t)|^{-1} + |y(t) - 1|^{-1} < \delta_1^{-1}\}.$$

Let the limit Stokes graph be as in Figure 4.1 (b). In what follows we use the following notation:

(1) for the WKB-solution of Proposition 3.4, write $\Lambda(\tau)$ in the component-wise form $\Lambda(\tau) = \Lambda_3(\tau) + \Lambda_I(\tau)$ with $\Lambda_3(\tau) \in \mathbb{C}\sigma_3$, $\Lambda_I(\tau) \in \mathbb{C}I$;

(2) $c_0 = (c_1 - ic_2)/c_3$ and $d_0 = (d_1 - id_2)/d_3$, where $c_k = b_k(\lambda_2)$, $d_k = b_k(\lambda_1)$ for $k = 1, 2, 3$.

In Propositions 3.4 and 3.5, set $\delta = \delta' = 1/4 - \varepsilon$, $0 < \varepsilon < 1/4$ being arbitrary. Then both propositions apply to the annulus

$$\mathcal{A}_\varepsilon : \quad |t|^{-\frac{2}{3} + \frac{2}{3}\left(\frac{1}{4} - \varepsilon\right)} \ll |\lambda - \lambda_\iota| \ll |t|^{-\frac{5}{12} - \frac{1}{3}\left(\frac{1}{4} - \varepsilon\right)} \ll |t|^{-\frac{1}{3} - \frac{1}{3}\left(\frac{1}{4} - \varepsilon\right)} \quad (\iota = 1, 2),$$

in the choice of which evaluation of $\eta^{1/2}$ in, say, **(b)** below is also taken into account.

Remark 4.1. The annulus $|t|^{-14/27 - (2/3)\varepsilon'} \ll |\lambda - \lambda_\iota| \ll |t|^{-14/27 + (1/3)\varepsilon'}$ of the early version, which is contained in $\mathcal{A}_\varepsilon$ with $\varepsilon = 1/36 + \varepsilon'$ for $\delta = \delta' = 2/9 - \varepsilon' = 1/4 - \varepsilon$, is also available in our calculation below, though the derivation of $-\frac{14}{27} - \frac{2}{3}\varepsilon'$, $-\frac{14}{27} + \frac{1}{3}\varepsilon'$ is based on an inaccurate argument.

In what follows we set $\delta = 1/4 - \varepsilon$.

Consider the Stokes graph on the plane $\mathbb{C} \setminus [e^{i\phi}, +\infty]$ containing the sector $0 < \arg\lambda < 2\pi$, $|\lambda| > 2025$. The connection matrix along $\gamma_2 = (-\hat{\mathbf{c}}_2^\infty) \cup \mathbf{c}_0 \cup \mathbf{c}_1^\infty$ consists of the following matrices $\Gamma_{\rm a}, \ldots, \Gamma_{\rm i}$.

(a) Let $\Psi_\infty(\lambda)$ be the WKB solution along $\mathbf{c}_1^\infty$ with a base point $\tilde{\lambda}_1 \in \mathbf{c}_1^\infty$, $|\tilde{\lambda}_1 - \lambda_1| \asymp t^{-1}$ near λ_1, and set $Y(\lambda) = \Psi_\infty(\lambda)\Gamma_{\rm a}$. Then we have

$$\begin{aligned}\Gamma_{\rm a} &= \exp\Big(-\int_{\tilde{\lambda}_1}^{\lambda} \Lambda(\tau)d\tau\Big) T^{-1}(I + O(t^{-\delta} + |\lambda|^{-1})) \exp\big(\tfrac14(t\lambda - 2\theta_\infty \log\lambda)\sigma_3\big)\\ &= C_3(\tilde{\lambda}_1)c_I(\tilde{\lambda}_1)(I + O(t^{-\delta})) \exp\Big(-\lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{c}_1^\infty}}\Big(\int_{\lambda_1}^{\lambda}\Lambda_3(\tau)d\tau - \frac14(t\lambda - 2\theta_\infty\log\lambda)\sigma_3\Big)\Big)\end{aligned}$$

with $C_3(\tilde{\lambda}_1) = \exp(\int_{\lambda_1}^{\tilde{\lambda}_1}\Lambda_3(\tau)d\tau)$, $c_I(\tilde{\lambda}_1) = \exp(-\int_{\tilde{\lambda}_1}^{\infty}\Lambda_I(\tau)d\tau)$.

(b) For $\Psi_\infty(\lambda)$ and $\Phi_1(\lambda)$ (cf. Proposition 3.5) in the annulus $\mathcal{A}_\varepsilon$ set $\Psi_\infty(\lambda) = \Phi_1(\lambda)\Gamma_{\rm b}$ along $\mathbf{c}_1^\infty$. We may suppose that the curve $(2\kappa)^{1/3}(\lambda - \tilde{\lambda}_1) = t^{-2/3}(\zeta + O(t^{-1/3}))$ with $\lambda \in \mathbf{c}_1^\infty$ enters the sector $\Sigma_1 : |\arg\zeta - \pi/3| < 2\pi/3$, and that Σ_1 does not intersect the cut $[e^{i\phi}, \lambda_1]$. Write $K^{-1} = 2(2\kappa)^{-1/3}(d_1 - id_2)$, where $d_k = b_k(\lambda_1)$ as in **(2)**. Then, by Propositions 3.4 and 3.5,

$$\begin{aligned}\Gamma_{\mathbf{b}} &= \Phi_1(\lambda)^{-1}\Psi_\infty(\lambda)\\ &= W(\zeta)^{-1}\begin{pmatrix}1 & 0\\ 0 & Kt^{-1/3}\end{pmatrix}^{-1}(I + O(t^{-\delta}))\begin{pmatrix}1 & -d_3/(d_1+id_2)\\ -d_3/(d_1-id_2) & 1\end{pmatrix}^{-1}\\ &\qquad\times\begin{pmatrix}1 & (b_3-\mu)/(b_1+ib_2)\\ (\mu-b_3)/(b_1-ib_2) & 1\end{pmatrix}(I + O(t^{-\delta}))\exp\Big(\int_{\tilde{\lambda}_1}^{\lambda}\Lambda(\tau)d\tau\Big)\\ &= W(\zeta)^{-1}\begin{pmatrix}1 & d_3/(d_1+id_2)\\ \mu t^{1/3}/(2K(d_1-id_2)) & \mu t^{1/3}/(2Kd_3)\end{pmatrix}(I + O(t^{-\delta}))\exp\Big(\int_{\tilde{\lambda}_1}^{\lambda}\Lambda(\tau)d\tau\Big)\end{aligned}$$

for $\lambda \in \mathcal{A}_\varepsilon \cap \mathbf{c}_2^\infty$, where $(\mu - b_3)/(b_1 \pm ib_2) = (\mu - d_3)/(d_1 \pm id_2) + O(\eta)$, $\eta = \lambda - \tilde{\lambda}_1$. Using

$$\begin{aligned}\mu &= (b_1^2 + b_2^2 + b_3^2)^{1/2} = (2(d_1d_1' + d_2d_2' + d_3d_3')(\eta + O(\eta^2)))^{1/2}\\ &= (2\kappa)^{1/2}\eta^{1/2}(1 + O(\eta)) = (2\kappa)^{1/3}t^{-1/3}\zeta^{1/2}(1 + O(\eta))\\ &= 2K(d_1 - id_2)t^{-1/3}\zeta^{1/2}(1 + O(\eta)),\end{aligned}$$

we have

$$\Gamma_{\rm b} = \exp\Big(\int_{\tilde{\lambda}_1}^{\lambda}\Lambda(\tau)d\tau - \frac23\zeta^{3/2}\sigma_3\Big)\zeta^{1/4}(I + O(t^{-\delta}))\begin{pmatrix}1 & 0\\ 0 & -d_0\end{pmatrix}.$$

By Remark 3.6, $\Lambda_3(\lambda) = [(2\kappa)^{1/2}t\eta^{1/2}(1 + O(\eta)) + O(\eta^{-1/2})]\sigma_3$ and $\Lambda_I(\lambda) = [-(\log\eta)_\eta/4 + O(\eta^{-1/2})]I$ for $\eta = \lambda - \tilde{\lambda}_1$, $\lambda \in \mathcal{A}_\varepsilon \cap \mathbf{c}_1^\infty$. Hence

$$\Gamma_{\rm b} = (I + O(t^{-\delta}))\exp\Big(-\int_{\lambda_1}^{\tilde{\lambda}_1}\Lambda_3(\tau)d\tau + O(\eta^{1/2})\Big)(\tilde{\zeta}_1)^{1/4}\begin{pmatrix}1 & 0\\ 0 & -d_0\end{pmatrix}$$

with suitably chosen $\tilde{\zeta}_1 \asymp \tilde{\eta}_1 = \tilde{\lambda}_1 - \lambda_1$. Since $\eta^{1/2} \ll t^{-1/4+\varepsilon/6} \ll t^{-\delta}$ in $\mathcal{A}_\varepsilon$,

$$\Gamma_{\mathrm{b}} = (\tilde{\zeta}_1)^{1/4}(I + O(t^{-\delta}))C_3(\tilde{\lambda}_1)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -d_0 \end{pmatrix}.$$

(c) Let $\Phi_{1-}(\lambda)$ be the solution by Proposition 3.5 near $\mathbf{c}_0$ in an annulus and set $\Phi_1(\lambda) = \Phi_{1-}(\lambda)G_{\mathrm{c}}$. Then, by Proposition 3.5,

$$G_{\mathrm{c}} = \Phi_{1-}(\lambda)^{-1}\Phi_1(\lambda) = (\Phi_1(\lambda)G_1)^{-1}\Phi_1(\lambda) = G_1^{-1} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}.$$

(d) Let $\Psi_{\infty 1}(\lambda)$ be the WKB solution along $\mathbf{c}_0$ with the base point $\tilde{\lambda}_1' \in \mathbf{c}_0$ near λ_1, and set $\Phi_{1-}(\lambda) = \Psi_{\infty 1}(\lambda)\Gamma_{\mathrm{d}}$. By the same argument as in step **(b)**, we have

$$\Gamma_{\mathrm{d}} = (\tilde{\zeta}_1')^{-1/4}(I + O(t^{-\delta}))\tilde{C}_3(\tilde{\lambda}_1')\begin{pmatrix} 1 & 0 \\ 0 & -d_0^{-1} \end{pmatrix}$$

with $\tilde{\zeta}_1' \asymp \tilde{\lambda}_1' - \lambda_1$, $\tilde{C}_3(\tilde{\lambda}_1') = \exp(\int_{\lambda_1}^{\tilde{\lambda}_1'} \Lambda_3(\tau)d\tau)$.

(e) Let $\Psi_{\infty 2}(\lambda)$ be the WKB solution along $\mathbf{c}_0$ with the base point $\lambda_2' \in \mathbf{c}_0$ near λ_2, and set $\Psi_{\infty 1}(\lambda) = \Psi_{\infty 2}(\lambda)\Gamma_{\mathrm{e}}$. Then

$$\Gamma_{\mathrm{e}} = (I + O(t^{-\delta}))\tilde{C}_3(\tilde{\lambda}_1')^{-1}\tilde{C}_3'(\lambda_2')\tilde{c}_I(\tilde{\lambda}_1', \lambda_2')\exp\Big(-\int_{\lambda_2}^{\lambda_1} \Lambda_3(\tau)d\tau\Big)$$

with $\tilde{c}_I(\tilde{\lambda}_1', \lambda_2') = \exp(\int_{\tilde{\lambda}_1'}^{\lambda_2'} \Lambda_I(\tau)d\tau)$, $\tilde{C}_3'(\lambda_2') = \exp(\int_{\lambda_2}^{\lambda_2'} \Lambda_3(\tau)d\tau)$.

(f) Let $\Phi_2(\lambda)$ be the solution by Proposition 3.5 along $\mathbf{c}_0$ near λ_2, and set $\Psi_{\infty 2}(\lambda) = \Phi_2(\lambda)\Gamma_{\mathrm{f}}$. Then

$$\Gamma_{\mathrm{f}} = (\zeta_2')^{1/4}(I + O(t^{-\delta}))\tilde{C}_3'(\tilde{\lambda}_2')^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -c_0 \end{pmatrix},$$

where $\zeta_2' \asymp \lambda_2' - \lambda_2$ and $c_k = b_k(\lambda_2)$.

(g) Let $\Phi_{2-}(\lambda)$ be the solution by Proposition 3.5 near $\hat{\mathbf{c}}_2^\infty$ in the annulus $\mathcal{A}_\varepsilon$ around λ_2, and set $\Phi_2(\lambda) = \Phi_{2-}(\lambda)G_{\mathrm{g}}$. Then

$$G_{\mathrm{g}} = \Phi_{2-}(\lambda)^{-1}\Phi_2(\lambda) = (\Phi_2(\lambda)G_0^{-1})^{-1}\Phi_2(\lambda) = G_0 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}.$$

(h) For the WKB-solution $\hat{\Psi}_\infty(\lambda)$ along $\hat{\mathbf{c}}_2^\infty$ set $\Phi_{2-}(\lambda) = \hat{\Psi}_\infty(\lambda)\Gamma_{\mathrm{h}}$ for $\lambda \in \mathcal{A}_\varepsilon \cap \hat{\mathbf{c}}_2^\infty$. Then

$$\Gamma_{\mathrm{h}} = (\tilde{\zeta}_2')^{-1/4}(I + O(t^{-\delta}))\hat{C}_3(\tilde{\lambda}_2')\begin{pmatrix} 1 & 0 \\ 0 & -c_0^{-1} \end{pmatrix}$$

with $\tilde{\zeta}_2' \asymp \tilde{\lambda}_2' - \lambda_2$, $\hat{C}_3(\tilde{\lambda}_2') = \exp\Big(\int_{\lambda_2}^{\tilde{\lambda}_2'} \Lambda_3(\tau)d\tau\Big)$.

(i) Set $\hat{\Psi}_\infty(\lambda) = Y_3(t,\lambda)\Gamma_{\mathrm{i}}$. Then

$$\begin{aligned}\hat{\Gamma}_\infty &= \hat{C}_3(\tilde{\lambda}_2')^{-1}\hat{c}_I(\tilde{\lambda}_2')(I + O(t^{-\delta})) \\ &\quad \times \exp\Big(\lim_{\substack{\lambda\to\infty \\ \lambda\in\hat{\mathbf{c}}_2^\infty}}\Big(\int_{\lambda_2}^{\lambda} \Lambda_3(\tau)d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log\lambda)\sigma_3\Big)\Big)\end{aligned}$$

with $\hat{c}_I(\tilde{\lambda}'_2)=\exp\Bigl(\int_{\tilde{\lambda}'_2}^{\infty}\Lambda_I(\tau)d\tau\Bigr)$.

Then, collecting the matrices above and using a symmetric property about the scalar part, we have

$$\begin{aligned}\Gamma^{\infty}_{\infty 2}&=\Gamma_{\rm i}\Gamma_{\rm h}G_{\rm g}\Gamma_{\rm f}\Gamma_{\rm e}\Gamma_{\rm d}G_{\rm c}\Gamma_{\rm b}\Gamma_{\rm a}\\ &=(I+O(t^{-\delta}))\exp(\hat{J}_2\sigma_3)\begin{pmatrix}1&0\\0&-c_0^{-1}\end{pmatrix}\begin{pmatrix}1&0\\-i&1\end{pmatrix}\begin{pmatrix}1&0\\0&-c_0\end{pmatrix}\\ &\quad\times\exp(-J_0\sigma_3)\begin{pmatrix}1&0\\0&-d_0^{-1}\end{pmatrix}\begin{pmatrix}1&i\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&-d_0\end{pmatrix}\exp(-J_1\sigma_3)\\ &=(I+O(t^{-\delta}))\begin{pmatrix}e^{\hat{J}_2-J_0-J_1}&-id_0e^{\hat{J}_2-J_0+J_1}\\ ic_0^{-1}e^{-\hat{J}_2-J_0-J_1}&e^{J_1-\hat{J}_2}(e^{J_0}+c_0^{-1}d_0e^{-J_0})\end{pmatrix}.\end{aligned}$$

Here

$$J_j=\lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{c}_j^{\infty}}}\Bigl(\int_{\lambda_j}^{\lambda}\Lambda_3(\tau)d\tau-\frac{1}{4}(t\lambda-2\theta_{\infty}\log\lambda)\sigma_3\Bigr)\quad(j=1,2),$$

$$\hat{J}_2=\lim_{\substack{\lambda\to\infty\\ \lambda\in\hat{\mathbf{c}}_2^{\infty}}}\Bigl(\int_{\lambda_2}^{\lambda}\Lambda_3(\tau)d\tau-\frac{1}{4}(t\lambda-2\theta_{\infty}\log\lambda)\sigma_3\Bigr),\quad J_0=J_2-J_1=\int_{\lambda_2}^{\lambda_1}\Lambda_3(\tau)d\tau.$$

The matrix $\Gamma^{\infty}_{\infty 1}$ is calculated by using the same Stokes graph of Figure 4.1 (b) on the plane $\mathcal{D}^0_+=\mathbb{C}\setminus[-\infty,-e^{i\phi}]$ containing the sector $-\pi<\arg\lambda<\pi$, $|\lambda|>2025$. Note that, in this case, the curve $\hat{\mathbf{c}}^{\infty}_2\subset\mathcal{D}^0_+$ tends to $e^{-\pi i/2}\infty$, and denote it by $\hat{\mathbf{c}}^{\infty}_{2*}$. The calculation of $\Gamma^{\infty}_{\infty 1}$ begins with setting $Y_1(\lambda)=\hat{\Psi}^*_{\infty}(\lambda)\Gamma^*_{\rm a}$, where $\hat{\Psi}^*_{\infty}(\lambda)$ is the WKB solution along $\hat{\mathbf{c}}^{\infty}_{2*}$ tending to $e^{-\pi i/2}\infty$. Note that $\hat{\Psi}^*_{\infty}(\lambda)$ has the same asymptotic form as of $\hat{\Psi}_{\infty}(\lambda)$. Repeating step by step matchings along $\gamma_1=(-\mathbf{c}^{\infty}_1)\cup\mathbf{c}_0\cup\hat{\mathbf{c}}^{\infty}_{2*}$, we have

$$\begin{aligned}\Gamma^{\infty}_{\infty 1}&=(I+O(t^{-\delta}))\exp(J_1\sigma_3)\begin{pmatrix}1&0\\0&-d_0^{-1}\end{pmatrix}\begin{pmatrix}1&-i\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&-d_0\end{pmatrix}\\ &\quad\times\exp(J_0\sigma_3)\begin{pmatrix}1&0\\0&-c_0^{-1}\end{pmatrix}\begin{pmatrix}1&0\\i&1\end{pmatrix}\begin{pmatrix}1&0\\0&-c_0\end{pmatrix}\exp(-\hat{J}^*_2\sigma_3)\\ &=(I+O(t^{-\delta}))\begin{pmatrix}e^{J_1-\hat{J}^*_2}(e^{J_0}+c_0^{-1}d_0e^{-J_0})&id_0e^{J_1-J_0+\hat{J}^*_2}\\ -ic_0^{-1}e^{-J_0-J_1-\hat{J}^*_2}&e^{\hat{J}^*_2-J_0-J_1}\end{pmatrix}\end{aligned}$$

with

$$\hat{J}^*_2=\lim_{\substack{\lambda\to\infty\\ \lambda\in\hat{\mathbf{c}}_{2*}^{\infty}}}\Bigl(\int_{\lambda_2}^{\lambda}\Lambda_3(\tau)d\tau-\frac{1}{4}(t\lambda-2\theta_{\infty}\log\lambda)\sigma_3\Bigr).$$

Thus we have $\Gamma^{\infty}_{\infty 2}$ and $\Gamma^{\infty}_{\infty 1}$ for $0<\phi<\pi/2$. In the case $-\pi/2<\phi<0$, using the Stokes graph of Figure 4.1 (a), we have similarly

$$\Gamma^{\infty}_{\infty 2}=(I+O(t^{-\delta}))\begin{pmatrix}e^{\hat{J}^*_1-J_2}(e^{J_0}+c_0^{-1}d_0e^{-J_0})&-id_0e^{J_2-J_0+\hat{J}^*_1}\\ ic_0^{-1}e^{-J_2-J_0-\hat{J}^*_1}&e^{J_2-J_0-\hat{J}^*_1}\end{pmatrix},$$

$$\Gamma^{\infty}_{\infty 1} = (I + O(t^{-\delta})) \begin{pmatrix} e^{J_2 - J_0 - \hat{J}_1} & i d_0 e^{-J_0 + J_2 + \hat{J}_1} \\ -i c_0^{-1} e^{-J_2 - J_0 - \hat{J}_1} & e^{-J_2 + \hat{J}_1}(e^{J_0} + c_0^{-1} d_0 e^{-J_0}) \end{pmatrix},$$

where

$$\hat{J}_1 = \lim_{\substack{\lambda \to \infty \\ \lambda \in \hat{\mathbf{c}}^{\infty}_1}} \Big(\int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3 \Big),$$

$$\hat{J}_1^* = \lim_{\substack{\lambda \to \infty \\ \lambda \in \hat{\mathbf{c}}^{\infty}_{1*}}} \Big(\int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3 \Big)$$

with a curve $\hat{\mathbf{c}}^{\infty}_{1*}$ tending to $e^{3\pi i/2}\infty$.

Let $M^0 = (m^0_{ij})$ and $M^1 = (m^1_{ij})$, and suppose that $0 < \phi < \pi/2$. The relations $\Gamma^{\infty}_{\infty 2} M^0_* = (S_2^*)^{-1}$, $M^1_* \Gamma^{\infty}_{\infty 1} = (S_1^*)^{-1}$ yield

$$\begin{aligned} & e^{\hat{J}_2 - J_0 - J_1} m^0_{11} - i d_0 e^{\hat{J}_2 - J_0 + J_1} \hat{u} m^0_{21} = 1, \\ & i c_0^{-1} e^{-\hat{J}_2 - J_0 - J_1} m^0_{11} + (e^{J_0} + c_0^{-1} d_0 e^{-J_0}) e^{-\hat{J}_2 + J_1} \hat{u} m^0_{21} = 0, \\ & i d_0 e^{\hat{J}_2^* - J_0 + J_1} m^1_{11} + e^{\hat{J}_2^* - J_0 - J_1} \hat{u}^{-1} m^1_{12} = 0, \\ & (e^{J_0} + c_0^{-1} d_0 e^{-J_0}) e^{-\hat{J}_2^* + J_1} m^1_{11} - i c_0^{-1} e^{-\hat{J}_2^* - J_0 - J_1} \hat{u}^{-1} m^1_{12} = 1 \end{aligned}$$

up to the factor $1 + O(t^{-\delta})$, from which it follows that

$$\begin{aligned} & \hat{u} m^0_{21} = -i c_0^{-1} e^{-\hat{J}_2 - J_0 - J_1}, \quad m^0_{11} = e^{J_1 - \hat{J}_2}(e^{J_0} + c_0^{-1} d_0 e^{-J_0}), \\ & \hat{u}^{-1} m^1_{12} = -i d_0 e^{-J_0 + J_1 + \hat{J}_2^*}, \quad m^1_{11} = e^{-J_0 - J_1 + \hat{J}_2^*}. \end{aligned}$$

Therefore entries of M^0 and M^1 satisfy

$$\frac{m^0_{11} m^1_{11}}{m^0_{21} m^1_{12}} = -(1 + c_0 d_0^{-1} e^{2J_0}), \quad m^0_{11} = e^{J_2 - \hat{J}_2}(1 + c_0^{-1} d_0 e^{-2J_0}).$$

Note that $m^0_{11} m^1_{11} + m^0_{21} m^1_{12} = e^{-\pi i \theta_\infty}$ by (3.2). From the first equation we have

$$e^{\pi i \theta_\infty} m^0_{21} m^1_{12} = -c_0^{-1} d_0 e^{-2J_0}(1 + O(t^{-\delta})),$$

and from the second equation

$$e^{J_2 - \hat{J}_2} = \frac{m^0_{11}(1 + O(t^{-\delta}))}{1 + c_0^{-1} d_0 e^{-2J_0}} = \frac{m^0_{11}(1 + O(t^{-\delta}))}{1 - m^0_{21} m^1_{12}(m^0_{11} m^1_{11} + m^0_{21} m^1_{12})^{-1}} = \frac{1}{e^{\pi i \theta_\infty} m^1_{11}}(1 + O(t^{-\delta})).$$

In the case $-\pi/2 < \phi < 0$, we have

$$\begin{aligned} & (e^{J_0} + c_0^{-1} d_0 e^{-J_0}) e^{\hat{J}_1^* - J_2} m^0_{11} - i d_0 e^{\hat{J}_1^* - J_0 + J_2} \hat{u} m^0_{21} = 1, \\ & i c_0^{-1} e^{-\hat{J}_1^* - J_0 - J_2} m^0_{11} + e^{-\hat{J}_1^* - J_0 + J_2} \hat{u} m^0_{21} = 0, \\ & e^{-\hat{J}_1 - J_0 + J_2} m^1_{11} - i c_0^{-1} e^{-\hat{J}_1 - J_0 - J_2} \hat{u}^{-1} m^1_{12} = 1, \\ & i d_0 e^{\hat{J}_1 - J_0 + J_2} m^1_{11} + (e^{J_0} + c_0^{-1} d_0 e^{-J_0}) e^{\hat{J}_1 - J_2} \hat{u}^{-1} m^1_{12} = 0 \end{aligned}$$

yielding

$$\hat{u} m^0_{21} = -i c_0^{-1} e^{-\hat{J}_1^* - J_0 - J_2}, \quad m^0_{11} = e^{-\hat{J}_1^* - J_0 + J_2},$$

$$\hat{u}^{-1}m^1_{12} = -id_0 e^{\hat{J}_1 - J_0 + J_2}, \quad m^1_{11} = (e^{J_0} + c_0^{-1} d_0 e^{-J_0}) e^{\hat{J}_1 - J_2},$$

from which we derive the first equality of the case above and $m^0_{11} = e^{J_2 - \hat{J}_2}(1 + O(t^{-\delta}))$. Then, for $0 < |\phi| < \pi/2$,

$$\mathfrak{m}_\phi = (1 + O(t^{-\delta})) e^{J_2 - \hat{J}_2},$$

$$\frac{m^0_{21} m^1_{12}}{m^0_{11} m^1_{11} + m^0_{21} m^1_{12}} = e^{\pi i \theta_\infty} m^0_{21} m^1_{12} = -(1 + O(t^{-\delta})) c_0^{-1} d_0 e^{2J_1 - 2J_2} \tag{4.4}$$

with $\mathfrak{m}_\phi$ as in Theorem 2.1. Here the contour of the integral $J_2 - \hat{J}_2$ on $\mathbb{C} \setminus [e^{i\phi}, +\infty]$ corresponds to the cycle $\mathbf{b}$ as in Section 5.1.

5. Asymptotics of monodromy data

To calculate the integrals $J_{1,2}$ and $\hat{J}_2$ we make a further change of variables

$$\lambda = \lambda(z) = e^{i\phi} z. \tag{5.1}$$

Proposition 5.1. *By* (5.1), *the turning points* $\lambda_1(t)$, $\lambda_2(t)$, $\lambda^0_1(t)$, $\lambda^0_2(t)$ *are mapped to*

$$z_1(t) = a_\phi^{1/2} + 2e^{-i\phi}\theta_\infty t^{-1} + O(t^{-2}), \qquad z_2(t) = -a_\phi^{1/2} + 2e^{-i\phi}\theta_\infty t^{-1} + O(t^{-2}),$$
$$z^0_1(t) = 1 + O(t^{-2}), \qquad z^0_2(t) = -1 + O(t^{-2}),$$

respectively.

By (5.1) the characteristic root (3.11) becomes

$$\mu = \mu(z) = \frac{1}{4}\sqrt{\frac{a_\phi - z^2}{1 - z^2}} + \frac{e^{-i\phi}\theta_\infty z}{2w} t^{-1} + \tilde{g}_2(t, z) t^{-2}, \tag{5.2}$$

where

$$w = w(z) = w(a_\phi, z) = \sqrt{(1 - z^2)(a_\phi - z^2)}$$

and $\tilde{g}_2(t, z) \ll 1$ if $|z^2 - 1|^{-1} + |z^2 - a_\phi|^{-1} \ll 1$. The image of $\mathbb{P} = \mathbb{P}_+ \cup \mathbb{P}_-$ under the map (5.1) is given by $\Pi^* = \Pi^*_+ \cup \Pi^*_-$ with $\Pi^*_\pm = z(\mathbb{P}_\pm)$ glued along the cuts $[z^0_2(t), z_2(t)]$ and $[z_1(t), z^0_1(t)]$, $\mathbb{P}$ being defined in Subsection 3.3. Then $\mu = \mu(z)$ is an algebraic function on Π^*. The elliptic curve defined by $w(z)$ is given by the two-sheeted Riemann surface $\Pi_{a_\phi} = \Pi_{a_\phi,+} \cup \Pi_{a_\phi,-}$ glued along the cuts $[-1, -a_\phi^{1/2}]$, $[a_\phi^{1/2}, 1]$ satisfying $z_{2,1}(t) = \mp a_\phi^{1/2} + O(t^{-1})$, $z^0_{2,1}(t) = \mp 1 + O(t^{-2})$. Each square root in (5.2) is such that $\sqrt{(a_\phi - z^2)/(1 - z^2)} \to 1$, $z^{-2} w(z) \to -1$ as $z \to \infty$ on $\Pi_{a_\phi,+}$, and then $a_\phi^{-1/2}\sqrt{(a_\phi - z^2)/(1 - z^2)}$, $a_\phi^{-1/2} w(z) \to 1$ as $z \to 0$ on $\Pi_{a_\phi,+}$. The elliptic curve $\Pi_{A_\phi} = \Pi_+ \cup \Pi_-$ of Section 2 is the image of $\mathbb{P}^\infty_+ \cup \mathbb{P}^\infty_- = \lim_{t\to\infty} \mathbb{P}_+ \cup \mathbb{P}_-$ under the map (5.1).

Recall the cycles $\mathbf{a}$ and $\mathbf{b}$ on Π_{A_ϕ} descried in Figure 2.2. We remark that $\mathbf{a}$ and $\mathbf{b}$ may also be regarded as those on Π_{a_ϕ} as in Figure 5.1, if t is sufficiently large and if the distance between $\mathbf{a} \cup \mathbf{b}$ and $\{\pm 1\} \cup \{\pm a_\phi^{1/2}\}$ is $\gg 1$. We use the same symbols $\mathbf{a}$ and $\mathbf{b}$ as in Figure 2.2, provided that $A_\phi^{1/2} = \lim_{t\to\infty} a_\phi^{1/2}(t)$, which will not cause confusions.

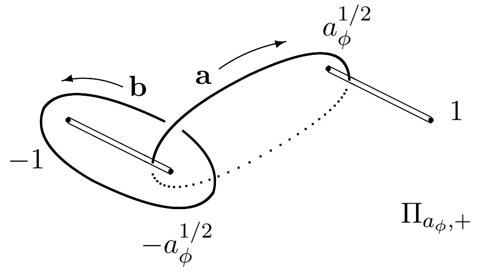


FIGURE 5.1. Cycles $\mathbf{a}$, $\mathbf{b}$ on Π_{a_ϕ}

5.1. **Expressions in terms of elliptic integrals.** Let J^μ_ι and $\hat{J}^\mu_\iota$ be such that $J^\mu_\iota\sigma_3 = J_\iota\sigma_3|_{\Lambda_3(\tau)\mapsto t\mu(\tau)\sigma_3}$ and $\hat{J}^\mu_\iota\sigma_3 = \hat{J}_\iota\sigma_3|_{\Lambda_3(\tau)\mapsto t\mu(\tau)\sigma_3}$, respectively. Note that $z_1(\infty) = A_\phi^{1/2}$, $z_2(\infty) = -A_\phi^{1/2}$, and that $z_{1,2}(\infty) - z_{1,2}(t) \ll t^{-1}$ by Proposition 5.1. Recall that $\lambda_{1,2} = \lambda_{1,2}(\infty)$ in these integrals. By use of $\int_{\lambda_\iota}^{\lambda}(t\mu(\tau) - \frac{1}{4}(t - 2\theta_\infty\tau^{-1}))d\tau$, we have

$$J^\mu_1 - J^\mu_2 = \int_{\lambda_1}^{\lambda_2} t\mu(\tau)d\tau = e^{i\phi}t\int_{A_\phi^{1/2}}^{-A_\phi^{1/2}} \mu(\lambda(z))dz = -e^{i\phi}t\int_{z_2(t)}^{z_1(t)} \mu(\lambda(z))dz + t(I_+ + I_-),$$

where

$$|I_\pm| \ll \left|\int_{\pm A_\phi^{1/2}}^{\pm a_\phi^{1/2} + 2e^{-i\phi}\theta_\infty t^{-1} + O(t^{-2})} \mu(\lambda(z))dz\right| \ll t^{-3/2}$$

with $A_\phi^{1/2} - a_\phi^{1/2} \ll t^{-1}$. Hence

$$\begin{aligned} J^\mu_1 - J^\mu_2 &= -\frac{e^{i\phi}}{2}t\int_{\mathbf{a}}\mu(\lambda(z))dz + O(t^{-1/2}) \\ &= -\frac{e^{i\phi}}{8}t\int_{\mathbf{a}}\Big(\sqrt{\frac{a_\phi - z^2}{1-z^2}} + \frac{2e^{-i\phi}\theta_\infty t^{-1}z}{w}\Big)dz - \frac{e^{i\phi}}{2}t^{-1}\int_{\mathbf{a}}\tilde{g}_2(t,z)dz + O(t^{-1/2}) \\ &= -\frac{e^{i\phi}}{8}t\int_{\mathbf{a}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz + O(t^{-1/2}). \end{aligned}$$

Furthermore we have

$$\begin{aligned} J^\mu_2 - \hat{J}^\mu_2 &= e^{i\phi}t\int_{\mathbf{b}}\mu(\lambda(z))dz + O(t^{-1/2}) \\ &= \frac{e^{i\phi}}{4}t\int_{\mathbf{b}}\Big(\sqrt{\frac{a_\phi - z^2}{1-z^2}} + \frac{2e^{-i\phi}\theta_\infty t^{-1}z}{w}\Big)dz + e^{i\phi}t^{-1}\int_{\mathbf{b}}\tilde{g}_2(t,z)dz + O(t^{-1/2}) \\ &= \frac{e^{i\phi}}{4}t\int_{\mathbf{b}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz - \frac{\theta_\infty\pi i}{2} + O(t^{-1/2}). \end{aligned}$$

The relation $\sqrt{(a_\phi - z^2)/(1-z^2)} = (1/w)(a_\phi - z^2) = -(w/z)' + a_\phi/w - z^{-2}a_\phi/w$ yields the following.

Proposition 5.2. *We have*

$$J^\mu_1 - J^\mu_2 = -\frac{e^{i\phi}}{8}t\int_{\mathbf{a}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz + O(t^{-1/2})$$

$$= -\frac{e^{i\phi}}{8}a_\phi t\int_{\mathbf{a}}\Big(\frac{1}{w}-\frac{1}{z^2w}\Big)dz + O(t^{-1/2}),$$

$$J_2^\mu - \hat{J}_2^\mu = \frac{e^{i\phi}}{4}t\int_{\mathbf{b}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz - \frac{\theta_\infty\pi i}{2} + O(t^{-1/2})$$

$$= \frac{e^{i\phi}}{4}a_\phi t\int_{\mathbf{b}}\Big(\frac{1}{w}-\frac{1}{z^2w}\Big)dz - \frac{\theta_\infty\pi i}{2} + O(t^{-1/2}).$$

To calculate $J_{1,2}$, $\hat{J}_2$ it is necessary to know the integrals of

$$\text{diag}T^{-1}T_\lambda|_{\sigma_3} = \frac{e^{-i\phi}}{4}\Big(1-\frac{b_3}{\mu}\Big)\frac{d}{dz}\log\frac{b_1+ib_2}{b_1-ib_2} = \frac{ie^{-i\phi}(b_1b_2'-b_1'b_2)}{2\mu(\mu+b_3)}. \tag{5.3}$$

By (5.1), b_k are written in the form

$$\begin{aligned}(z^2-1)b_3 &= \frac{1}{4}(z^2-1-4\mathfrak{z}_0)+O(t^{-1}),\\ y(z^2-1)(b_1+ib_2) &= \frac{1}{2}(y-1)\mathfrak{z}_0(z+1) - y\mathfrak{z}_0 + O(t^{-1}),\\ (z^2-1)(b_1-ib_2) &= \frac{1}{2}(y-1)\mathfrak{z}_0(z+1)+\mathfrak{z}_0+O(t^{-1})\end{aligned} \tag{5.4}$$

with $\mathfrak{z}_0 = -(e^{-i\phi}y^* - y)(y-1)^{-2}$. Let $z_\pm$ be such that

$$b_1(z_+)+ib_2(z_+)=0, \quad b_1(z_-)-ib_2(z_-)=0.$$

It is easy to see that

$$z_+ = \frac{y+1}{y-1}+O(t^{-1}), \quad z_- = -\frac{y+1}{y-1}+O(t^{-1}), \tag{5.5}$$

and that $\mu(z_\pm)^2 = b_3(z_\pm)^2$. By (5.3), $\|\text{diag}T^{-1}T_\lambda\| \ll |z\mp 1|^{-1/2}$ near $z=\pm1$. Furthermore, $\text{diag}T^{-1}T_\lambda$ has poles at $z_\pm \in \Pi_-$ and is holomorphic around $z_\pm\in\Pi_+$. From (3.9) combined with (5.2) it follows that

$$(1-z^2)\mu = \frac{1}{4}w(z)\Big(1+\frac{2e^{-i\phi}\theta_\infty z t^{-1}}{a_\phi - z^2}+O(t^{-2})\Big).$$

Note that $b_3(z_\pm) = e^{-i\phi}y^{-1}y^*/4+O(t^{-1})$ and $\mu(z_\pm)^2 = e^{-2i\phi}y^{-2}(y^*)^2/16+O(t^{-1})$. When $z_\pm\in\Pi_-$,

$$((z_\pm)^2-1)b_3(z_\pm) = -((z_\pm)^2-1)\mu(z_\pm) = \frac{1}{4}w(z_\pm)(1+O(t^{-1})). \tag{5.6}$$

The relations

$$\begin{aligned}(z^2-1)b_3\Big(\frac{1}{z-z_+}-\frac{1}{z-z_-}\Big) =& \frac{1}{4}(z_+-z_-)+\frac{((z_+)^2-1)b_3(z_+)}{z-z_+}-\frac{((z_-)^2-1)b_3(z_-)}{z-z_-}\\ =&\frac{1}{4}\Big(z_+-z_-+\frac{w(z_+)}{z-z_+}-\frac{w(z_-)}{z-z_-}\Big)+O(t^{-1})\end{aligned}$$

and

$$\text{diag}T^{-1}T_\lambda|_{\sigma_3} = \frac{e^{-i\phi}}{4}\Big(1-\frac{b_3}{\mu}\Big)\Big(\frac{1}{z-z_+}-\frac{1}{z-z_-}\Big)$$

yield

$$\mathrm{diag}T^{-1}T_\lambda|_{\sigma_3} = \frac{e^{-i\phi}}{4}\Big(\frac{1}{z-z_+} - \frac{1}{z-z_-} + \Big(z_+ - z_- + \frac{w(z_+)}{z-z_+} - \frac{w(z_-)}{z-z_-}\Big)\frac{1}{w(z)} + O(t^{-1})\Big).$$

Hence we have

$$\Big(\int_{\mathbf{c}_1^\infty} - \int_{\mathbf{c}_2^\infty}\Big)\mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}d\tau = \int_{\lambda_1}^{\lambda_2}\mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}d\tau = e^{i\phi}\int_{z_1}^{z_2}\mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}dz$$
$$= \frac{1}{4}\log\frac{(z_2-z_+)(z_1-z_-)}{(z_2-z_-)(z_1-z_+)}$$
$$- \frac{1}{8}\int_{\mathbf{a}}\Big(\frac{z_+-z_-}{w} + \frac{w(z_+)}{(z-z_+)w} - \frac{w(z_-)}{(z-z_-)w}\Big)dz + \pi i r_{\mathbf{a}} + O(t^{-1}),$$

and

$$\Big(\int_{\mathbf{c}_2^\infty} - \int_{\hat{\mathbf{c}}_2^\infty}\Big)\mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}d\tau = e^{i\phi}\int_{\mathbf{b}}\mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}dz$$
$$= \frac{1}{4}\int_{\mathbf{b}}\Big(\frac{z_+-z_-}{w} + \frac{w(z_+)}{(z-z_+)w} - \frac{w(z_-)}{(z-z_-)w}\Big)dz + 2\pi i r_{\mathbf{b}} + O(t^{-1}),$$

where $\pi i r_{\mathbf{a}}$, $2\pi i r_{\mathbf{b}}$ with $r_{\mathbf{a}}, r_{\mathbf{b}} = 0, 1$ are the contributions from the poles $z_\pm$ in deforming the contours. Since $(b_1(z) - ib_2(z))/(b_1(z) + ib_2(z)) = y(z-z_-)/(z-z_+)$,

$$c_0^2 = -\frac{c_1 - ic_2}{c_1 + ic_2} = -\frac{y(z_2-z_-)}{z_2-z_+}, \quad d_0^2 = -\frac{y(z_1-z_-)}{z_1-z_+}.$$

These combined with Proposition 5.2 and (4.4) yield the following.

Proposition 5.3. *Set*

$$W_0(z) = \frac{e^{i\phi}}{4}a_\phi\Big(\frac{1}{w} - \frac{1}{z^2 w}\Big), \quad W_1(z) = \frac{1}{4}\Big(\frac{z_+-z_-}{w} + \frac{w(z_+)}{(z-z_+)w} - \frac{w(z_-)}{(z-z_-)w}\Big)$$

with $w = w(z) = w(a_\phi, z)$. *Then*

$$J_2 - \hat{J}_2 = \frac{e^{i\phi}}{4}t\int_{\mathbf{b}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz - \int_{\mathbf{b}}W_1(z)dz - \frac{\theta_\infty\pi i}{2} - 2\pi i r_{\mathbf{b}} + O(t^{-1/2})$$
$$= \int_{\mathbf{b}}(tW_0(z) - W_1(z))dz - \frac{\theta_\infty\pi i}{2} - 2\pi i r_{\mathbf{b}} + O(t^{-1/2}),$$
$$2(J_1 - J_2) + \log(c_0^{-1}d_0) = -\frac{e^{i\phi}}{4}t\int_{\mathbf{a}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz + \int_{\mathbf{a}}W_1(z)dz + 2\pi i r_{\mathbf{a}} + O(t^{-1/2})$$
$$= -\int_{\mathbf{a}}(tW_0(z) - W_1(z))dz + 2\pi i r_{\mathbf{a}} + O(t^{-1/2}).$$

Corollary 5.4. *We have*

$$\log\mathfrak{m}_\phi = \frac{e^{i\phi}}{4}t\int_{\mathbf{b}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz - \int_{\mathbf{b}}W_1(z)dz - \frac{\theta_\infty\pi i}{2} + O(t^{-\delta})$$
$$= \int_{\mathbf{b}}(tW_0(z) - W_1(z))dz - \frac{\theta_\infty\pi i}{2} + O(t^{-\delta}),$$
$$\log(m_{21}^0 m_{12}^1) = -\frac{e^{i\phi}}{4}t\int_{\mathbf{a}}\sqrt{\frac{a_\phi - z^2}{1-z^2}}dz + \int_{\mathbf{a}}W_1(z)dz - (\theta_\infty + 1)\pi i + O(t^{-\delta})$$

$$= -\int_{\mathbf{a}} (tW_0(z) - W_1(z))dz - (\theta_\infty + 1)\pi i + O(t^{-\delta}).$$

5.2. **Expressions in terms of the ϑ-function.** Under the supposition $a_\phi \ne 0, 1$, write

$$\check{\mathrm{sn}}(u) = a_\phi^{1/2} \mathrm{sn}(u; a_\phi^{1/2}).$$

Then $z = \check{\mathrm{sn}}(u)$ satisfies $(dz/du)^2 = w(z)^2 = (1-z^2)(a_\phi - z^2)$. Setting $z = \check{\mathrm{sn}}(u)$ we have, for a given z_0 on the elliptic curve $\Pi_{a_\phi} = \Pi_{a_\phi,+} \cup \Pi_{a_\phi,-}$ for $w(z) = w(a_\phi, z)$,

$$\frac{dz}{(z-z_0)w(z)} = \frac{du}{\check{\mathrm{sn}}(u) - \check{\mathrm{sn}}(u_0)}, \quad z_0 = \check{\mathrm{sn}}(u_0).$$

Let $u_0^\pm$ be such that $z_0^{(\pm)} = \check{\mathrm{sn}}(u_0^\pm)$, where $z_0^{(+)} = (z_0, w(z_0^{(+)}))$, $z_0^{(-)} = (z_0, -w(z_0^{(+)}))$. Since $\check{\mathrm{sn}}'(u_0^\pm) = \pm w(z_0^{(+)})$,

$$\begin{aligned}\frac{1}{\check{\mathrm{sn}}(u) - z_0} &= \frac{1}{w(z_0^{(+)})}(\zeta(u-u_0^+) - \zeta(u-u_0^-)) - \frac{1}{w(z_0^{(+)})}\Big(\frac{w'(z_0^{(+)})}{2} - \zeta(u_0^+ - u_0^-)\Big) \\ &= \frac{1}{w(z_0^{(+)})}\frac{d}{du}\log\frac{\sigma(u-u_0^+)}{\sigma(u-u_0^-)} - \frac{1}{w(z_0^{(+)})}\Big(\frac{w'(z_0^{(+)})}{2} - \frac{\sigma'}{\sigma}(u_0^+ - u_0^-)\Big).\end{aligned}$$

This function may be written in terms of the ϑ-function

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi i z n}, \quad \mathrm{Im}\,\tau > 0$$

coinciding with ϑ_3 of Jacobi and having the properties:

$$\vartheta(z\pm 1,\tau) = \vartheta(z,\tau), \quad \vartheta(-z,\tau) = \vartheta(z,\tau), \quad \vartheta(z\pm\tau,\tau) = e^{-\pi i(\tau\pm 2z)}\vartheta(z,\tau)$$

[14, 43]. Let us write the fundamental periods of Π_{a_ϕ} as

$$\omega_{\mathbf{a}} = \omega_{\mathbf{a}}(t) = \int_{\mathbf{a}} \frac{dz}{w(z)}, \quad \omega_{\mathbf{b}} = \omega_{\mathbf{b}}(t) = \int_{\mathbf{b}} \frac{dz}{w(z)}, \quad w(z) = w(a_\phi, z).$$

Then $\check{\mathrm{sn}}(u)$ with the modulus $k = a_\phi^{1/2}$ has the periods $\omega_{\mathbf{a}} = 4K$, $\omega_{\mathbf{b}} = 2iK'$, and

$$\begin{aligned} d\log\frac{\sigma(u-u_0^+)}{\sigma(u-u_0^-)} &= \frac{2\zeta(\omega_{\mathbf{a}}/2)}{\omega_{\mathbf{a}}}(u_0^- - u_0^+)du + d\log\frac{\vartheta(F(z_0^{(+)},z)+\nu,\tau)}{\vartheta(F(z_0^{(-)},z)+\nu,\tau)}, \\ \frac{\sigma'}{\sigma}(u_0^+ - u_0^-) &= -\frac{2\zeta(\omega_{\mathbf{a}}/2)}{\omega_{\mathbf{a}}}(u_0^- - u_0^+) + \frac{i\pi}{\omega_{\mathbf{a}}} + \frac{1}{\omega_{\mathbf{a}}}\frac{\vartheta'}{\vartheta}(F(z_0^{(-)}, z_0^{(+)}) + \nu, \tau)\end{aligned}$$

[14, 43], where

$$\tau = \frac{\omega_{\mathbf{b}}}{\omega_{\mathbf{a}}}, \quad \nu = \frac{1}{2}(1+\tau), \quad F(z_*, z) = \frac{1}{\omega_{\mathbf{a}}}\int_{z_*}^{z}\frac{dz}{w(z)} = \frac{1}{\omega_{\mathbf{a}}}(u - u_*) \tag{5.7}$$

with $z = \check{\mathrm{sn}}(u)$, $z_* = \check{\mathrm{sn}}(u_*)$. Thus we have

$$\begin{aligned}\frac{dz}{(z-z_0)w(z)} &= \frac{1}{w(z_0^{(+)})} d\log\frac{\vartheta(F(z_0^{(+)},z)+\nu,\tau)}{\vartheta(F(z_0^{(-)},z)+\nu,\tau)} \\ &\quad - \frac{1}{w(z_0^{(+)})}\Big(\frac{w'(z_0^{(+)})}{2} - \frac{1}{\omega_{\mathbf{a}}}\Big(i\pi + \frac{\vartheta'}{\vartheta}(F(z_0^{(-)}, z_0^{(+)}) + \nu, \tau)\Big)\Big)\frac{dz}{w(z)},\end{aligned}$$

which yields

$$\int_{\mathbf{a}} \frac{dz}{(z-z_0)w(z)} = -\frac{w'(z_0^{(+)})}{2w(z_0^{(+)})}\omega_{\mathbf{a}} + \frac{1}{w(z_0^{(+)})}\Big(i\pi + \frac{\vartheta'}{\vartheta}(F(z_0^{(-)}, z_0^{(+)}) + \nu, \tau)\Big), \tag{5.8}$$

$$\int_{\mathbf{b}} \frac{dz}{(z-z_0)w(z)} = \frac{2\pi i}{w(z_0^{(+)})}F(z_0^{(-)}, z_0^{(+)}) + \tau \int_{\mathbf{a}} \frac{dz}{(z-z_0)w(z)}. \tag{5.9}$$

The integrals

$$\int_{\mathbf{a}} \frac{dz}{z^2 w(z)} = \frac{1}{2}(1+a_\phi^{-1})\omega_{\mathbf{a}} - \frac{2}{a_\phi \omega_{\mathbf{a}}}\Big(\frac{\vartheta''}{\vartheta} - \Big(\frac{\vartheta'}{\vartheta}\Big)^2\Big)(\tau/2, \tau),$$

$$\int_{\mathbf{b}} \frac{dz}{z^2 w(z)} = \frac{4\pi i}{a_\phi \omega_{\mathbf{a}}} + \tau \int_{\mathbf{a}} \frac{dz}{z^2 w(z)}$$

follow from $(\partial/\partial z_0)(5.8)$, $(\partial/\partial z_0)(5.9)$ with $z_0 = 0$. Furthermore, we obtain

$$\int_{\mathbf{a}} \frac{w(z_0)dz}{(z-z_0)w(z)} = -z_0\omega_{\mathbf{a}} + 2\frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_0^{(-)}, z_0^{(+)}) - \frac{1}{4}, \tau\Big),$$

combining (5.8) with the relation

$$\Big(\frac{z_0}{2} - \frac{w'(z_0)}{4}\Big)\omega_{\mathbf{a}} + \frac{1}{2}\frac{\vartheta'}{\vartheta}(F(z_0^{(-)}, z_0^{(+)}) + \nu, \tau) + \frac{\pi i}{2} = \frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_0^{(-)}, z_0^{(+)}) - \frac{1}{4}, \tau\Big),$$

which is derived by comparing the poles $z_0 = \pm 1$, $\pm a_\phi^{1/2}$ and ∞^+ ($\in \Pi_{a_\phi,+}$) on Π_{a_ϕ} (cf. [22, p.513], [28, (3.5)]).[2] In what follows let us adopt the convention that the path of the integral $\omega_{\mathbf{a}} F(z_0^{(-)}, z_0^{(+)}) = \int_{z_0^{(-)}}^{z_0^{(+)}} w(z)^{-1}dz$ passes through $a_\phi^{1/2}$, i.e. the left end of the cut $[a_\phi^{1/2}, 1]$. Then

$$\begin{aligned}\int_{z_0^{(-)}}^{z_0^{(+)}} \frac{dz}{w(z)} =& 2\int_{a_\phi^{1/2}}^{z_0^{(+)}} \frac{dz}{w(z)} = -2\int_{-a_\phi^{1/2}}^{-z_0^{(+)}} \frac{dz}{w(z)} \\ =& -2\Big(\int_{a_\phi^{1/2}}^{-z_0^{(+)}} + \int_{-a_\phi^{1/2}}^{a_0^{1/2}}\Big)\frac{dz}{w(z)} = -\int_{-z_0^{(-)}}^{-z_0^{(+)}} \frac{dz}{w(z)} - \omega_{\mathbf{a}},\end{aligned}$$

and hence $-F(-z_0^{(-)}, -z_0^{(+)}) = F(z_0^{(-)}, z_0^{(+)}) + 1$. From these relations with $z_0 = z_+$, z_- such that $z_+ + z_- = O(t^{-1})$ (cf. (5.5)), we derive the following.

Proposition 5.5. *For $W_0(z)$ and $W_1(z)$ in Proposition 5.3,*

$$\int_{\mathbf{a}} W_0(z)dz = \frac{e^{i\phi}}{8}\Big((a_\phi - 1)\omega_{\mathbf{a}} + \frac{4}{\omega_{\mathbf{a}}}\Big(\frac{\vartheta''}{\vartheta} - \Big(\frac{\vartheta'}{\vartheta}\Big)^2\Big)(\tau/2, \tau)\Big),$$

$$\int_{\mathbf{b}} W_0(z)dz - \tau\int_{\mathbf{a}} W_0(z)dz = -\frac{e^{i\phi}\pi i}{\omega_{\mathbf{a}}},$$

$$\int_{\mathbf{a}} W_1(z)dz = \frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_+^{(-)}, z_+^{(+)}) - \frac{1}{4}, \tau\Big) + O(t^{-1}),$$

[2] For $z_0 = \infty^+$ by using $\int_0^\infty w(z)^{-1}dz = \omega_{\mathbf{b}}/2$, it is shown that $F(z_0^{(-)}, z_0^{(+)}) = \tau - 1/2$ and the residues of the poles on both sides coincide. Similarly for each of $z_0 = \pm 1$ and $z_0 = \pm a_\phi^{1/2}$ the poles on the left-hand side are cancelled out, since $F(z_0^{(-)}, z_0^{(+)}) = 2\sqrt{2}(1-a_\phi)^{-1/2}(\omega_{\mathbf{a}})^{-1}t(1+o(1))$, say, for $z_0 = 1 + t^2 \to 1$. The difference of both sides is a constant, and setting $z_0 = 0$ we obtain this relation.

$$\int_{\mathbf{b}} W_1(z)dz - \tau \int_{\mathbf{a}} W_1(z)dz = \pi i F(z_+^{(-)}, z_+^{(+)}) + \frac{\pi i}{2} + O(t^{-1}),$$

where $z_+ = (y+1)/(y-1) + O(t^{-1})$, $z_+^{(+)} = (z_+, w(z_+^{(+)}))$, $z_+^{(-)} = (z_+, -w(z_+^{(+)}))$.

5.3. **Expression of** $B_\phi(t)$**.** Recall that $a_\phi(t) = A_\phi + t^{-1}B_\phi(t)$, $B_\phi(t) = O(1)$ in the domain $S_*(\phi, t'_\infty, \kappa_0, \delta_1)$, where A_ϕ is a unique solution of the Boutroux equations (2.3). The quantity $B_\phi(t)$ is written in terms of

$$\Omega_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}} \frac{dz}{w(A_\phi, z)}, \quad \mathcal{E}_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}} \sqrt{\frac{A_\phi - z^2}{1-z^2}} dz,$$

where $w(A_\phi, z) = \sqrt{(A_\phi - z^2)(1-z^2)}$, and $\mathbf{a}$, $\mathbf{b}$ are basic cycles on Π_{A_ϕ}. Observing that

$$\sqrt{\frac{a_\phi - z^2}{1-z^2}} - \sqrt{\frac{A_\phi - z^2}{1-z^2}} = \frac{1}{\sqrt{1-z^2}}(\sqrt{a_\phi - z^2} - \sqrt{A_\phi - z^2}) = \frac{t^{-1}B_\phi}{2w(A_\phi, z)}(1 + O(t^{-1}B_\phi)),$$

and combining this with Corollary 5.4 and Proposition 5.5, we obtain

$$\frac{e^{i\phi}}{4}\Big(t\mathcal{E}_{\mathbf{a}} + \frac{\Omega_{\mathbf{a}}}{2}B_\phi(1 + O(t^{-1}B_\phi))\Big) = \int_{\mathbf{a}} W_1(z)dz - \log(m_{21}^0 m_{12}^1) - \pi i(\theta_\infty + 1) + O(t^{-\delta})$$

with $\int_{\mathbf{a}} W_1(z)dz$ as in Proposition 5.5.

Proposition 5.6. *In* $S_*(\phi, t'_\infty, \kappa_0, \delta_1)$, $B_\phi(t)$ *is bounded, and*

$$\frac{e^{i\phi}}{4}\Big(t\mathcal{E}_{\mathbf{a}} + \frac{\Omega_{\mathbf{a}}}{2}B_\phi\Big) = \frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_+^{(-)}, z_+^{(+)}) - \frac{1}{4}, \tau\Big) - \log(m_{21}^0 m_{12}^1) - \pi i(\theta_\infty + 1) + O(t^{-\delta}).$$

Remark 5.1. In the argument above, the substitution $(\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{b}, -\mathbf{a})$ yields

$$\frac{e^{i\phi}}{4}\Big(t\mathcal{E}_{\mathbf{b}} + \frac{\Omega_{\mathbf{b}}}{2}B_\phi(1 + O(t^{-1}B_\phi))\Big) = \int_{\mathbf{b}} W_1(z)dz + \frac{\theta_\infty}{2}\pi i + \log \mathfrak{m}_\phi + O(t^{-\delta})$$

with

$$\int_{\mathbf{b}} W_1(z)dz = \frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}\hat{F}(z_+^{(-)}, z_+^{(+)}) + \frac{\hat{\tau}}{4}, \hat{\tau}\Big) + O(t^{-1}),$$

in which $\hat{F}$ denotes F corresponding to $\hat{\tau} = (-\omega_{\mathbf{a}})/\omega_{\mathbf{b}}$. Since $\mathrm{Re} \int_{\mathbf{a},\mathbf{b}} W_1(z)dz$ are bounded in $S_*(\phi, t'_\infty, \kappa_0, \delta_1)$, the Boutroux equations (2.3) with A_ϕ are equivalent to the boundedness of $\mathrm{Re}\,(e^{i\phi}\Omega_{\mathbf{a}}B_\phi)$ and $\mathrm{Re}\,(e^{i\phi}\Omega_{\mathbf{b}}B_\phi)$, namely, the boundedness of $B_\phi(t)$.

Proposition 5.7. *We have*

$$\int_{z_+^{(-)}}^{z_+^{(+)}} \frac{dz}{w(A_\phi, z)} = \int_{z_+^{(-)}}^{z_+^{(+)}} \frac{dz}{w(z)} + O(t^{-1}), \quad w(z) = w(a_\phi, z).$$

To show this proposition, we note the following lemma, which is verified by combining

$$3w(A_\phi, z) = (zw(A_\phi, z))' + (A_\phi + 1)\sqrt{\frac{A_\phi - z^2}{1-z^2}} - A_\phi(A_\phi - 1)\frac{1}{w(A_\phi, z)}$$

with

$$J_{\mathbf{a}}\Omega_{\mathbf{b}} - J_{\mathbf{b}}\Omega_{\mathbf{a}} = \frac{4}{3}(1 + A_\phi)\pi i, \quad J_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}} w(A_\phi, z)dz.$$

The derivation of the last equality is similar to that of Legendre's relation [14, 43].

Lemma 5.8. $\mathcal{E}_{\mathbf{a}}\Omega_{\mathbf{b}} - \mathcal{E}_{\mathbf{b}}\Omega_{\mathbf{a}} = 4\pi i$.

Proof. By the boundedness of $B_\phi(t)$, $\omega_{\mathbf{a},\mathbf{b}} = \Omega_{\mathbf{a},\mathbf{b}} + O(t^{-1})$ for $\mathbf{a}$, $\mathbf{b} \subset \Pi_{a_\phi} \cap \Pi_{A_\phi}$. From Proposition 5.5 and Corollary 5.4, it follows that

$$\Big(\int_{\mathbf{b}} - \tau \int_{\mathbf{a}}\Big)(tW_0(z) - W_1(t))dz = -\frac{e^{i\phi}\pi i}{\omega_{\mathbf{a}}}t - \pi i F(z_+^{(-)}, z_+^{(+)}) - \frac{\pi i}{2} + O(t^{-1})$$
$$= -\frac{e^{i\phi}\pi i}{\omega_{\mathbf{a}}}t - 2\pi i\Big(p_+(t) + \frac{\omega_{\mathbf{b}}}{\omega_{\mathbf{a}}}q_+(t)\Big) + O(1) \ll 1$$

with $p_+(t)$, $q_+(t) \in \mathbb{Z}$. Write $e^{i\phi}t\mathcal{E}_{\mathbf{a}}/8 + \pi i q_+(t) = Xi$ and $e^{i\phi}t\mathcal{E}_{\mathbf{b}}/8 - \pi i p_+(t) = Yi$, where, by (2.3), $|\mathrm{Im}\, X|, |\mathrm{Im}\, Y| \ll 1$. Then, by Lemma 5.8, the last line becomes

$$-\frac{e^{i\phi}\pi i}{\Omega_{\mathbf{a}}}t - \frac{e^{i\phi}}{4}\Big(\mathcal{E}_{\mathbf{b}} - \frac{\Omega_{\mathbf{b}}}{\Omega_{\mathbf{a}}}\mathcal{E}_{\mathbf{a}}\Big)t - 2\Big(\frac{\omega_{\mathbf{b}}}{\omega_{\mathbf{a}}}X - Y\Big)i + O(1) = -2\Big(\frac{\omega_{\mathbf{b}}}{\omega_{\mathbf{a}}}X - Y\Big)i + O(1) \ll 1,$$

where $\mathrm{Im}\,(\omega_{\mathbf{b}}/\omega_{\mathbf{a}}) \to \mathrm{Im}\,(\Omega_{\mathbf{b}}/\Omega_{\mathbf{a}}) > 0$ as $t \to \infty$. This implies $|X|$, $|Y| \ll 1$, and hence

$$\pi i q_+(t) = -e^{i\phi}\mathcal{E}_{\mathbf{a}}t/8 + O(1), \quad \pi i p_+(t) = e^{i\phi}\mathcal{E}_{\mathbf{b}}t/8 + O(1) \tag{5.10}$$

as $t \to \infty$. We would like to evaluate

$$\Upsilon = \Big|\int_{z_+^{(-)}}^{z_+^{(+)}} \Big(\frac{1}{w(z)} - \frac{1}{w(A_\phi, z)}\Big)dz\Big|,$$

in which the integrand is

$$\frac{1}{w(z)} - \frac{1}{w(A_\phi, z)} = \frac{-B_\phi(t)t^{-1}}{2(A_\phi - z^2)w(A_\phi, z)} + O(t^{-2}).$$

Observe that the contour $[z_+^{(-)}, z_+^{(+)}]^\sim$ on $\Pi_{a_\phi} \cap \Pi_{A_\phi}$ may be decomposed into $2p_+(t)\mathbf{a} \cup 2q_+(t)\mathbf{b} \cup \mathbf{a}_+ \cup \mathbf{a}_-$, where the length of $\mathbf{a}_\pm$ is $\ll 1$. Using (5.10) and Lemma 5.8, we have

$$\Upsilon \ll \Big|\int_{z_+^{(-)}}^{z_+^{(+)}} \frac{B_\phi(t)t^{-1}}{(A_\phi - z^2)w(A_\phi, z)}dz\Big| + O(t^{-1}) \ll |B_\phi t^{-1}||p_+(t)j_{\mathbf{a}} + q_+(t)j_{\mathbf{b}}| + O(t^{-1})$$
$$\ll |\mathcal{E}_{\mathbf{b}}j_{\mathbf{a}} - \mathcal{E}_{\mathbf{a}}j_{\mathbf{b}}| + O(t^{-1}) = 2\Big|\frac{\partial}{\partial A_\phi}(\mathcal{E}_{\mathbf{a}}\Omega_{\mathbf{b}} - \mathcal{E}_{\mathbf{b}}\Omega_{\mathbf{a}})\Big| + O(t^{-1}) \ll t^{-1}$$

with

$$j_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}} \frac{dz}{(A_\phi - z^2)w(A_\phi, z)},$$

which completes the proof of the proposition. □

6. Proofs of Theorems 2.1 and 2.2

Let $y(t)$ be a function satisfying (4.3), and let $(M^0, M^1) \in \mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$ be such that $\mathfrak{m}_\phi m_{21}^0 m_{12}^1 \neq 0$. Suppose that $0 < |\phi| < \pi/2$.

6.1. **Proof of Theorem 2.1.** Note that, by our convention,

$$F(z_+^{(-)}, z_+^{(+)}) = \frac{1}{\omega_{\mathbf{a}}}\int_{z_+^{(-)}}^{z_+^{(+)}} \frac{dz}{w(z)} = \frac{2}{\omega_{\mathbf{a}}}\int_{0^{(+)}}^{z_+^{(+)}} \frac{dz}{w(z)} - \frac{1}{2} + O(t^{-1}) \tag{6.1}$$

on Π_{a_ϕ}. By Propositions 5.5, 5.7 and Corollary 5.4,

$$\begin{aligned}
(6.2)\qquad & \omega_{\mathbf{a}}\Big(\int_{\mathbf{b}} - \tau \int_{\mathbf{a}}\Big)(tW_0(z) - W_1(z))dz \\
&= \omega_{\mathbf{a}}\Big(\log \mathfrak{m}_\phi + \tau(\log(m_{21}^0 m_{12}^1) + \pi i(\theta_\infty + 1)) + \frac{\pi i \theta_\infty}{2} + O(t^{-\delta})\Big) \\
&= \Omega_{\mathbf{a}}\Big(\log \mathfrak{m}_\phi + \frac{\Omega_{\mathbf{b}}}{\Omega_{\mathbf{a}}}(\log(m_{21}^0 m_{12}^1) + \pi i(\theta_\infty + 1)) + \frac{\pi i \theta_\infty}{2} + O(t^{-\delta})\Big) \\
&= -e^{i\phi}\pi i t - \pi i \omega_{\mathbf{a}}(F(z_+^{(-)}, z_+^{(+)}) + \tfrac{1}{2}) + O(t^{-\delta}) \\
&= -e^{i\phi}\pi i t - 2\pi i \int_{0^{(+)}}^{z_+^{(+)}} \frac{dz}{w(z)} + O(t^{-\delta}) \\
&= -e^{i\phi}\pi i t - 2\pi i \int_{0^{(+)}}^{z_{+0}^{(+)}} \frac{dz}{w(A_\phi, z)} + O(t^{-\delta})
\end{aligned}$$

with $z_{+0}^{(+)} = (y(t)+1)/(y(t)-1)$. From (6.2) with $\omega_{\mathbf{a},\mathbf{b}} = \Omega_{\mathbf{a},\mathbf{b}} + O(t^{-1})$, it follows that

$$\int_{0^{(+)}}^{z_{+0}^{(+)}} \frac{dz}{w(A_\phi, z)} = -\frac{1}{2}(e^{i\phi}t - \tilde{x}_0) + O(t^{-\delta}),$$

where $\tilde{x}_0 = x_0 + \Omega_{\mathbf{a}}$ and

$$x_0 \equiv \frac{-1}{\pi i}\Big(\Omega_{\mathbf{b}} \log(m_{21}^0 m_{12}^1) + \Omega_{\mathbf{a}} \log \mathfrak{m}_\phi\Big) - \Big(\frac{\Omega_{\mathbf{a}}}{2} + \Omega_{\mathbf{b}}\Big)(\theta_\infty + 1) - \frac{\Omega_{\mathbf{a}}}{2} \mod 2\Omega_{\mathbf{a}}\mathbb{Z} + 2\Omega_{\mathbf{b}}\mathbb{Z}.$$

This gives

$$\frac{y(t)+1}{y(t)-1} = A_\phi^{1/2}\mathrm{sn}((e^{i\phi}t - x_0)/2 + O(t^{-\delta}); A_\phi^{1/2}) \tag{6.3}$$

as $t \to \infty$ through $S_*(\phi, t'_\infty, \kappa_0, \delta_1)$. Thus we obtain the desired asymptotic form.

Let $W_1^*(z)$ be the result of replacement of $w(z)$ with $w(A_\phi, z)$ in $W_1(z)$, that is,

$$W_1^*(z) = \frac{1}{4}\Big(\frac{z_+ - z_-}{w(A_\phi, z)} + \frac{w(A_\phi, z_+)}{(z - z_+)w(A_\phi, z)} - \frac{w(A_\phi, z_-)}{(z - z_-)w(A_\phi, z)}\Big),$$

which differs from the early $W_1(z)$ by $O(t^{-1})$ along $\mathbf{a}$, since $B_\phi(t) \ll 1$. Then, by Proposition 5.5

$$\int_{\mathbf{a}} W_1(z) = \int_{\mathbf{a}} W_1^*(z)dz + O(t^{-1}) = \frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F^*(z_+^{(-)}, z_+^{(+)}) - \frac{1}{4}, \frac{\Omega_{\mathbf{b}}}{\Omega_{\mathbf{a}}}\Big) + O(t^{-1})$$

with

$$F^*(z_+^{(-)}, z_+^{(+)}) = \frac{1}{\Omega_{\mathbf{a}}}\int_{z_+^{(-)}}^{z_+^{(+)}} \frac{dz}{w(A_\phi, z)} = \frac{2}{\Omega_{\mathbf{a}}}\int_{0^{(+)}}^{z_+^{(+)}} \frac{dz}{w(A_\phi, z)} - \frac{1}{2} + O(t^{-1}).$$

The same argument as in the derivation of Proposition 5.6 leads the following.

Corollary 6.1. *In* $S_*(\phi, t'_\infty, \kappa_0, \delta_0)$,

$$\frac{e^{i\phi}}{4}\Big(t\mathcal{E}_{\mathbf{a}} + \frac{\Omega_{\mathbf{a}}}{2}B_\phi\Big) = -\frac{\vartheta'}{\vartheta}\Big(\frac{1}{2\Omega_{\mathbf{a}}}(e^{i\phi}t - x_0), \frac{\Omega_{\mathbf{b}}}{\Omega_{\mathbf{a}}}\Big) - \log(m^0_{21}m^1_{12}) - \pi i(\theta_\infty + 1) + O(t^{-\delta}).$$

Justification The justification of (6.3) as a solution of ($\mathrm{P_V}$) is carried out along the argument in [28, pp. 105–106, pp. 120–121]. Let $\mathcal{M} = (M^0, M^1) \in \mathcal{M}_{(\theta_0,\theta_1,\theta_\infty)}$ be such that $\mathfrak{m}_\phi m^0_{21}m^1_{12} \ne 0$ (cf. Remark 2.2). Relation (6.3) and Corollary 6.1 provide the leading terms $y_{\mathrm{as}} = y_{\mathrm{as}}(\mathcal{M}, t)$ and $(B_\phi)_{\mathrm{as}} = (B_\phi)_{\mathrm{as}}(\mathcal{M}, t)$ without $O(t^{-\delta})$. Viewing (4.3), we set $2y^*_{\mathrm{as}} = 2y^*_{\mathrm{as}}(\mathcal{M}, t) = \sqrt{\varphi(t, y_{\mathrm{as}}, (B_\phi)_{\mathrm{as}})}$ with

$$\begin{aligned}\varphi(t, y, B) =& e^{2i\phi}y(4y + (1 - A_\phi)(y - 1)^2) \\ &+ e^{i\phi}y(y - 1)(4(\theta_0 + \theta_1)(y + 1) - e^{i\phi}(y - 1)B)t^{-1} \\ &+ (y - 1)^3((\theta_0 - \theta_1 + \theta_\infty)^2y - (\theta_0 - \theta_1 - \theta_\infty)^2)t^{-2},\end{aligned}$$

where the branch of the square root is chosen in such a way that the leading term of y^*_{as} is compatible with $(d/dt)y_{\mathrm{as}}$. Then $(y_{\mathrm{as}}, y^*_{\mathrm{as}}) = (y_{\mathrm{as}}(\mathcal{M}, t), y^*_{\mathrm{as}}(\mathcal{M}, t))$ satisfies (4.3) with $B_\phi(t) = (B_\phi)_{\mathrm{as}}(\mathcal{M}, t)$ in $\hat{S}(\phi, t_\infty, \kappa_0, \delta_2)$, and the relation $\mathcal{M} = \mathcal{M}(t, y_{\mathrm{as}}, y^*_{\mathrm{as}})$ [28, (2.28)]. Here $\hat{S}(\phi, t_\infty, \kappa_0, \delta_2) = \{t \mid \mathrm{Re}\, t > t_\infty, |\mathrm{Im}\, t| < \kappa_0\} \setminus \bigcup_{\sigma\in Z_\infty\cup Z_0}\{|t - e^{-i\phi}\sigma| \le \delta_2\}$ with $Z_\infty = \{e^{i\phi}t \mid y_{\mathrm{as}}(t) = 1\} = x_0 + \Omega_{\mathbf{a}}\mathbb{Z} + 2\Omega_{\mathbf{b}}(\mathbb{Z} + \frac12)$, $Z_0 = \{e^{i\phi}t \mid y_{\mathrm{as}}(t) = 0, \infty\} = x_0 + \Omega_{\mathbf{a}}(\mathbb{Z} + \frac12) + 2\Omega_{\mathbf{b}}(\mathbb{Z} + \frac12)$, and $\mathcal{M}(t, y, y^*)$ is a collection of explicit functions (t, y, y^*) resulting from the WKB procedure.[3] The monodromy data $\mathcal{M}_{\mathrm{as}}(t)$ for system (3.4) with $\mathcal{B}(t, \lambda)$ containing $(y_{\mathrm{as}}, y^*_{\mathrm{as}})$ is given by $\mathcal{M}_{\mathrm{as}}(t) = \mathcal{M}(t, y_{\mathrm{as}}, y^*_{\mathrm{as}}) + O(t^{-\delta})$ [28, (2.27)] as a result of the repeated WKB procedure. Thus, for $|t| \ge T_0$, we have $\|\mathcal{M}_{\mathrm{as}}(t) - \mathcal{M}\| \le Ct^{-\delta}$ valid uniformly in a neighbourhood of $\mathcal{M}$, where C and T_0 are independent of $\mathcal{M}$ [28, (2.26)]. Then the justification scheme [26] applies to our case (see also [12, Theorem 5.5]). This justification combined with the maximal modulus principle in each neighbourhood of $\sigma \in Z_0$ leads us Theorem 2.1 in $S(\rho, t_\infty, \kappa_0, \delta_0)$.

The justification scheme is described as follows. Let $\mathcal{M}_0$ be a given monodromy data, and $K(\varepsilon_0) : \|\mathcal{M} - \mathcal{M}_0\| \le \varepsilon_0$ a given compact ball with centre at $\mathcal{M}_0$. By the property above combined with the compactness, there exists positive numbers T_∞ and C_0 such that, for every $\mathcal{M} \in K(\varepsilon_0)$, $\|\mathcal{M} - (\mathcal{M})_{\mathrm{as}}\| \le C_0|t|^{-\delta}$ if $|t| \ge T_\infty$. Note that $f(t, \mathcal{M}) := \mathcal{M}_0 - (\mathcal{M})_{\mathrm{as}}(t) + \mathcal{M}$ is a map f: $K(\varepsilon_0) \to K(\varepsilon_0)$ continuous in $\mathcal{M} \in K(\varepsilon_0)$ if $\|\mathcal{M} - (\mathcal{M})_{\mathrm{as}}(t)\| \le \varepsilon_0$, i.e. if $|t|^\delta \ge C_0/\varepsilon_0$. Then by Brouwer's fixed point theorem, for each $|t| \ge \max\{T_\infty, (C_0/\varepsilon_0)^{1/\delta}\}$, there exists a fixed point $\mathcal{M}_* = \mathcal{M}_*(t) \in K(\varepsilon_0)$ such that $f(t, \mathcal{M}_*) = \mathcal{M}_*$ i.e. $(\mathcal{M}_*)_{\mathrm{as}}(t) = \mathcal{M}_0$. (As will be shown later $\mathcal{M}_*$ is a unique fixed point.) This implies that $\|\mathcal{M}_* - \mathcal{M}_0\| = \|\mathcal{M}_* - (\mathcal{M}_*)_{\mathrm{as}}(t)\| \le C_0|t|^{-\delta}$ and hence

[3] In our case $(y_{\mathrm{as}}, y^*_{\mathrm{as}})$ may be replaced with $(y_{\mathrm{as}}, (B_\phi)_{\mathrm{as}})$, and then $\mathcal{M}(t, y_{\mathrm{as}}, y^*_{\mathrm{as}})$ consists of, say,
$-e^{i\phi}\pi it - \pi i\Omega_{\mathbf{a}}(F^*(z^{(-)}_{\mathrm{as}}, z^{(+)}_{\mathrm{as}}) + \frac12) - \Omega_{\mathbf{a}}\pi i(\frac32\theta_\infty + 1)$,
$\frac14 e^{i\phi}(t\mathcal{E}_{\mathbf{a}} + \frac12\Omega_{\mathbf{a}}(B_\phi)_{\mathrm{as}}) - (\vartheta'/\vartheta)(\frac12 F^*(z^{(-)}_{\mathrm{as}}, z^{(+)}_{\mathrm{as}}) - \frac14, \Omega_{\mathbf{b}}/\Omega_{\mathbf{a}}) + \pi i(\theta_\infty + 1)$,
$z^{(+)}_{\mathrm{as}} = (y_{\mathrm{as}} + 1)/(y_{\mathrm{as}} - 1)$,
which are main parts of (6.2) and of the equation of Proposition 5.6 without $O(t^{-\delta})$.

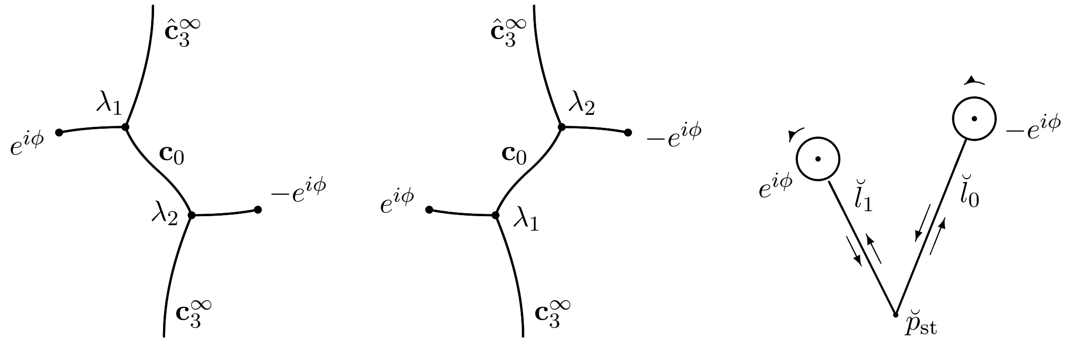
$\mathcal{M}_* = \mathcal{M}_0 + O(t^{-\delta})$ for $|t| \ge \max\{T_\infty, (C_0/\varepsilon_0)^{1/\delta}\}$. Then $(\mathcal{M}_*)_{\rm as}(t) = \mathcal{M}_0$ leads us to

$$\begin{aligned}(y_{\rm as}(\mathcal{M}_*, t), (B_\phi)_{\rm as}(\mathcal{M}_*, t)) &= (y_{\rm as}(\mathcal{M}_0 + O(t^{-\delta}), t), (B_\phi)_{\rm as}(\mathcal{M}_0 + O(t^{-\delta}), t)) \\ &= (y_{\rm as}(\mathcal{M}_0, t + O(t^{-\delta})), (B_\phi)_{\rm as}(\mathcal{M}_0, t + O(t^{-\delta}))),\end{aligned}$$

which realises isomonodromy deformation with the invariant monodromy data $\mathcal{M}_0$, and then $y^*_{\rm as} = (d/dt)y_{\rm as}$. This provides the desired expression $y_{\rm as}(\mathcal{M}_0, t + O(t^{-\delta}))$ in our main result. It remains to show the uniqueness of $\mathcal{M}_*$. To do so suppose that there exist $\mathcal{M}_{*1}$ and $\mathcal{M}_{*2}$ such that $\mathcal{M}_0 = (\mathcal{M}_{*1})_{\rm as}(t) = (\mathcal{M}_{*2})_{\rm as}(t)$, and set $y_1 = y_{\rm as}(\mathcal{M}_{*1}, t)$, $y_2 = y_{\rm as}(\mathcal{M}_{*2}, t)$. Let $V_1(\lambda)$ and $V_2(\lambda)$ be matrix solutions solving (3.4) with $\mathcal{B}(t,\lambda)$ containing (y_1, y_1') and (y_2, y_2'), respectively. Since $V_1(\lambda)$ and $V_2(\lambda)$ yield the same monodromy, by the uniqueness of Proposition 3.2 we have $(dV_1/d\lambda)V_1^{-1} = (dY^{(2)}/d\lambda)V_2^{-1}$, implying $y_1 = y_2$, which contradicts the supposition.

Thus Theorem 2.1 is proved.

6.2. **Proof of Theorem 2.2.** Suppose that $0 < |\phi - \pi| < \pi/2$, i.e. $\pi/2 < \phi < 3\pi/2$. Recall that $\mu(\lambda)$ is on the Riemann surface $\mathbb{P}_+ \cup \mathbb{P}_-$ glued along the cuts $[\lambda_1^0, \lambda_1]$ and $[\lambda_2, \lambda_2^0]$. Note that $A_\phi = A_{\phi-\pi}$ by Lemma 8.14. Let $\mathcal{S}(\pi/2, 3\pi/2)$ on $\mathbb{P}_+^\infty$ be the limit Stokes graph as described in Figure 6.1 (a), (b), in which $\hat{\mathbf{c}}_3^\infty$ joins λ_1 or λ_2 to $e^{5\pi i/2}\infty$, and $\mathbf{c}_3^\infty$ joins λ_2 or λ_1 to $e^{3\pi i/2}\infty$. The anticlockwise π-radian rotation of the Stokes graph $\mathcal{S}(-\pi/2, \pi/2)$ for $0 < |\phi| < \pi/2$ as in Figure 4.1 results in $\mathcal{S}(\pi/2, 3\pi/2)$. The curve $\mathbf{c}_3^\infty$ corresponds to $\mathbf{c}_1^\infty$ or $\mathbf{c}_2^\infty$. Let the loops $\breve{l}_0$, $\breve{l}_1$ be the results of the same rotation of $\hat{l}_0$, $\hat{l}_1$ in Figure 3.1. The loops $\breve{l}_0$, $\breve{l}_1$ and the starting point $\breve{p}_{\rm st}$ are as in Figure 6.1 (c), and $\arg(\breve{p}_{\rm st}) = 3\pi/2$. Let $\breve{M}^0$ and $\breve{M}^1$ be the monodromy matrices defined by the



(a) $\pi/2 < \phi < \pi$ (b) $\pi < \phi < 3\pi/2$ (c) Loops $\breve{l}_0$ and $\breve{l}_1$

FIGURE 6.1. Limit Stokes graph and loops

analytic continuation of $Y_3(t,\lambda)$ along the loops $\breve{l}_0$ and $\breve{l}_1$, respectively. Recalling that $Y_3(t,\lambda) = Y(t,\lambda)S_2$, and that the analytic continuation of $Y(t,\lambda) = Y_2(t,\lambda)$ along $\hat{l}_0$, $\hat{l}_1$ are $Y(t,\lambda)M^0$, $Y(t,\lambda)M^1$, respectively, we have $S_2^{-1}M^0S_2 = \breve{M}^0$, $S_2^{-1}M^1S_2 = \breve{M}^1$.

In the calculation of $\breve{M}^0$ and $\breve{M}^1$ this Stokes graph is used. Suppose that $\pi < \phi < 3\pi/2$. Let $Y_4(t,\lambda)$ be the matrix solution admitting the same asymptotic representation as (3.6) in the sector $|\arg\lambda - 5\pi/2| < \pi$. Denote by Γ_3 a connection matrix such that $Y_3 = Y_4\Gamma_3$ along $(-\hat{\mathbf{c}}_3^\infty) \cup \mathbf{c}_0 \cup \mathbf{c}_3^\infty$ joining $e^{5\pi i/2}\infty$ to $e^{3\pi i/2}\infty$. The Stokes matrix

$S_3 = e^{\pi i\theta_\infty\sigma_3}S_1e^{-\pi i\theta_\infty\sigma_3}$ is given by $Y_4 = Y_3S_3$. Then $\Gamma_3\breve{M}^0 = S_3^{-1}$. The WKB analysis with the Stokes graph in Figure 6.1 (b) on $\mathbb{C}\setminus[-\infty, e^{i\phi}]$ leads us to

$$\Gamma_3 = (I + O(t^{-\delta}))\begin{pmatrix} e^{\hat{J}_3-J_3}(e^{-J_0} + c_0d_0^{-1}e^{J_0}) & ic_0e^{J_0+\hat{J}_3+J_3} \\ -id_0^{-1}e^{J_0-\hat{J}_3-J_3} & e^{J_0-\hat{J}_3+J_3}\end{pmatrix},$$

where $J_0 = \int_{\lambda_2}^{\lambda_1}\Lambda_3(\tau)d\tau$,

$$J_3 = \lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{c}_3^\infty}}\Big(\int_{\lambda_1}^{\lambda}\Lambda_3(\tau)d\tau - \tfrac{1}{4}(t\lambda - 2\theta_\infty\log\lambda)\sigma_3\Big),$$

$$\hat{J}_3 = \lim_{\substack{\lambda\to\infty\\ \lambda\in\hat{\mathbf{c}}_3^\infty}}\Big(\int_{\lambda_2}^{\lambda}\Lambda_3(\tau)d\tau - \tfrac{1}{4}(t\lambda - 2\theta_\infty\log\lambda)\sigma_3\Big).$$

Recall $\Gamma_{\infty 2}^\infty$ such that $Y_3\Gamma_{\infty 2}^\infty = Y = Y_2$ along a path joining $e^{3\pi i/2}\infty$ to $e^{\pi i/2}\infty$ on the right-hand side of $e^{i\phi}$ in Figure 6.1 (b). Then $\breve{M}^1\Gamma_{\infty 2}^\infty = S_2^{-1}$, and, by the use of the Stokes graph on $\mathbb{C}\setminus[-e^{i\phi}, +\infty]$ with $\mathbf{c}_2^\infty$ joining λ_2 to $e^{\pi i/2}\infty$ in place of $\hat{\mathbf{c}}_3^\infty$,

$$\Gamma_{\infty 2}^\infty = (I + O(t^{-\delta}))\begin{pmatrix} e^{J_3-J_2+J_0} & -ic_0e^{J_0+J_3+J_2} \\ id_0^{-1}e^{J_0-J_3-J_2} & e^{-J_3+J_2}(e^{-J_0} + c_0d_0^{-1}e^{J_0})\end{pmatrix}$$

with J_2 as in Section 4. Note that $\breve{m}_{12}^0\breve{m}_{21}^1 + \breve{m}_{22}^0\breve{m}_{22}^1 = e^{\pi i\theta_\infty}$ follows from $\breve{M}^1\breve{M}^0 = S_2^{-1}S_1^{-1}e^{-\pi i\theta_\infty\sigma_3}$. From the relations $\Gamma_3\breve{M}^0 = S_3^{-1}$ and $\breve{M}^1\Gamma_{\infty 2}^\infty = S_2^{-1}$, it follows that

$$\breve{m}_{12}^0 = -ic_0e^{\hat{J}_3+J_3+J_0}, \quad \breve{m}_{22}^0 = e^{\hat{J}_3-J_3-J_0}(1 + c_0d_0^{-1}e^{2J_0}),$$
$$\breve{m}_{22}^1 = e^{J_3+J_0-J_2}, \quad \breve{m}_{21}^1 = -id_0^{-1}e^{-J_3+J_0-J_2}$$

up to the factor $1 + O(t^{-\delta})$, which implies $\breve{m}_{22}^0\breve{m}_{22}^1(\breve{m}_{12}^0\breve{m}_{21}^1)^{-1} = -1 - c_0^{-1}d_0e^{-2J_0}$. Then we derive $-c_0^{-1}d_0e^{-2J_0}(1 + O(t^{-\delta})) = e^{\pi i\theta_\infty}(\breve{m}_{12}^0\breve{m}_{21}^1)^{-1}$, and

$$e^{J_3+J_0-\hat{J}_3} = \frac{1 + c_0d_0^{-1}e^{2J_0}}{\breve{m}_{22}^0} = \frac{-1 - c_0^{-1}d_0e^{-2J_0}}{\breve{m}_{22}^0\,(-c_0^{-1}d_0e^{-2J_0})} = e^{-\pi i\theta_\infty}\breve{m}_{22}^1,$$

in which the contour of $J_3 + J_0 - \hat{J}_3$ on $\mathbb{C}\setminus[-\infty, e^{i\phi}]$ corresponds to the cycle $\mathbf{b}$. These relations leads to the conclusion for $\pi < \phi < 3\pi/2$. In the case $\pi/2 < \phi < \pi$, denoting $J_3|_{\lambda_1\mapsto\lambda_2}$ and $\hat{J}_3|_{\lambda_2\mapsto\lambda_1}$ by the same symbols J_3 and $\hat{J}_3$, respectively, we have, by using the Stokes graph in Figure 6.1 (a),

$$\Gamma_3 = (I + O(t^{-\delta}))\begin{pmatrix} e^{J_0+\hat{J}_3-J_3} & ic_0e^{J_0+\hat{J}_3+J_3} \\ -id_0^{-1}e^{J_0-\hat{J}_3-J_3} & e^{-\hat{J}_3+J_3}(e^{-J_0} + c_0d_0^{-1}e^{J_0})\end{pmatrix},$$

$$\Gamma_{\infty 2}^\infty = (I + O(t^{-\delta}))\begin{pmatrix} e^{J_3-J_1}(e^{-J_0} + c_0d_0^{-1}e^{J_0}) & -ic_0e^{J_0+J_3+J_1} \\ id_0^{-1}e^{J_0-J_3-J_1} & e^{-J_3+J_1+J_0}\end{pmatrix}.$$

These relation yields

$$e^{J_3-J_0-\hat{J}_3}(1 + O(t^{-\delta})) = (\breve{m}_{22}^0)^{-1}, \quad -c_0^{-1}d_0e^{-2J_0}(1 + O(t^{-\delta})) = e^{\pi i\theta_\infty}(\breve{m}_{12}^0\breve{m}_{21}^1)^{-1},$$

the first of which immediately follows from $\Gamma_3\breve{M}^0 = S_3^{-1}$. From these relations the phase shift $\breve{x}_0$ is derived by a procedure analogous to that for x_0. Thus Theorem 2.2 is obtained.

For ϕ such that $|\phi - k\pi| < \pi/2$ ($k \in \mathbb{Z}$), denote by $\hat{l}_0^{(k)}$, $\hat{l}_1^{(k)}$ and $\mathcal{S}(k\pi - \pi/2, k\pi + \pi/2)$ the $k\pi$-rotation of $\hat{l}_0$, $\hat{l}_1$ and $\mathcal{S}(-\pi/2, \pi/2)$, respectively. Let $\tilde{Y}^p(t,\lambda)$ be the canonical solution of (3.4) admitting the same form as of (3.6) in the sector $|\arg\lambda - 2p\pi - \pi/2| < \pi$, and let M_p^0 and M_p^1 be the monodromy matrices given by the analytic continuations of $\tilde{Y}^p(t,\lambda)$ along $\hat{l}_0^{(2p)}$ and $\hat{l}_1^{(2p)}$, respectively. Especially, $\tilde{Y}^0(t,\lambda) = Y(t,\lambda)$, $\hat{l}_0^{(0)} = \hat{l}_0$, $\hat{l}_1^{(0)} = \hat{l}_1$, $M_0^0 = M^0$, $M_0^1 = M^1$. Then M_p^0 and M_p^1 are as in Remark 2.5, and M_p^0, M_p^1 are calculated on $\mathcal{S}(2p\pi - \pi/2, 2p\pi + \pi/2)$. Thus this is reduced to the situation, to which Theorem 2.1 applies. For the canonical solution $\breve{Y}^p(t,\lambda)$ in the sector $|\arg\lambda - (2p+1)\pi - \pi/2| < \pi$ such that $\breve{Y}^0(t,\lambda) = Y_3(t,\lambda)$, the analytic continuations along $\hat{l}_0^{(2p+1)}$ and $\hat{l}_1^{(2p+1)}$ yield $\breve{M}_p^0$ and $\breve{M}_p^1$ as in Remark 2.5. Calculation of $\breve{M}_p^0$ and $\breve{M}_p^1$ on $\mathcal{S}((2p+1)\pi - \pi/2, (2p+1)\pi + \pi/2)$ leads to the results corresponding to Theorem 2.2 (cf. Remark 2.5).

7. System equivalent to ($\mathrm{P_V}$) and another approach to $B_\phi(t)$

Multiplying both sides of ($\mathrm{P_V}$) by $2(dy/dx)y^{-1}(y-1)^{-2}$, we write ($\mathrm{P_V}$) in the form

$$\frac{d}{dx}L = -2x^{-1}L - \frac{2x^{-1}y}{(y-1)^2} + 2(1-\theta_0-\theta_1)\frac{x^{-2}}{y-1}, \tag{7.1}$$

where

$$\begin{aligned} L = L(x) :=& \frac{(y')^2}{y(y-1)^2} - \frac{y}{(y-1)^2} + 2(1-\theta_0-\theta_1)\frac{x^{-1}}{y-1} \\ &- \frac{x^{-2}}{4}\Big((\theta_0-\theta_1+\theta_\infty)^2 y + (\theta_0-\theta_1-\theta_\infty)^2\frac{1}{y}\Big). \end{aligned}$$

Furthermore $L = L(x)$ is written in terms of $\psi := (y+1)/(y-1)$:

$$\begin{aligned} L(x) =& \frac{(\psi')^2}{\psi^2-1} - \frac{1}{4}(\psi^2-1) - (1-\theta_0-\theta_1)x^{-1}(1-\psi) \\ &+ \frac{x^{-2}}{4}\Big((\theta_0-\theta_1+\theta_\infty)^2\frac{1+\psi}{1-\psi} + (\theta_0-\theta_1-\theta_\infty)^2\frac{1-\psi}{1+\psi}\Big) \end{aligned} \tag{7.2}$$

with $\psi' = d\psi/dx$. Then (7.1) equivalent to ($\mathrm{P_V}$) becomes

$$\frac{d}{dx}L = -2x^{-1}L - \frac{1}{2}(\psi^2-1)x^{-1} + (\theta_0+\theta_1-1)(1-\psi)x^{-2}. \tag{7.3}$$

The quantity a_ϕ defined by (3.10) with $y^* = y_t(t) = e^{i\phi}y'(x)$ is rewritten in the form

$$a_\phi = 1 - 4\frac{(y')^2-y^2}{y(y-1)^2} + 4(\theta_0+\theta_1)x^{-1}\frac{y+1}{y-1} + x^{-2}\frac{y-1}{y}((\theta_0-\theta_1+\theta_\infty)^2 y - (\theta_0-\theta_1-\theta_\infty)^2),$$

and then

$$\begin{aligned} 4(\psi')^2 =& (1-\psi^2)(a_\phi-\psi^2) - 4(\theta_0+\theta_1)x^{-1}\psi(1-\psi^2) \\ &+ 4x^{-2}(2(\theta_0-\theta_1)\theta_\infty\psi + (\theta_0-\theta_1)^2 + \theta_\infty^2). \end{aligned} \tag{7.4}$$

From (7.2) and (7.4) it follows that

$$L = \frac{1}{4}(1 - a_\phi) + (\theta_0 + \theta_1 - 1 + \psi)x^{-1} - \frac{1}{2}((\theta_0 - \theta_1)^2 + \theta_\infty^2)x^{-2}. \tag{7.5}$$

The system consisting of (7.3) and (7.4) with (7.5) may be regarded as one with respect to ψ and $a_\phi = A_\phi + x^{-1}b(x)$ that is equivalent to ($\mathrm{P_V}$). The system of equations (7.3) and (7.4) is also written in the form

$$\begin{aligned} 4(\psi')^2 =&(1 - \psi^2)(A_\phi - \psi^2) - (1 - \psi^2)(4(\theta_0 + \theta_1)\psi - b)x^{-1} \\ &+ 4(2(\theta_0 - \theta_1)\theta_\infty\psi + (\theta_0 - \theta_1)^2 + \theta_\infty^2)x^{-2}, \end{aligned} \tag{7.6}$$

$$b' = -2(A_\phi - \psi^2) + 4\psi' + (4(\theta_0 + \theta_1)\psi - b)x^{-1}, \tag{7.7}$$

which follows from the substitution $a_\phi(x) \mapsto A_\phi + x^{-1}b(x)$ in (7.3) and (7.4) with (7.5). Neglecting the terms with the multiplier x^{-1} in (7.6) and (7.7), we have

$$4(\tilde{\psi}')^2 = (1 - \tilde{\psi}^2)(A_\phi - \tilde{\psi}^2), \quad \tilde{b}' = -2(A_\phi - \tilde{\psi}^2) + 4\tilde{\psi}'.$$

The first equation admits the solution

$$\psi_0(x) = A_\phi^{1/2}\mathrm{sn}((x - x_0)/2; A_\phi^{1/2}), \quad 4(\psi_0')^2 = (1 - \psi_0^2)(A_\phi - \psi_0^2)$$

expressed by the Jacobi sn-function with $\Omega_{\mathbf{a}} = 4K$, $\Omega_{\mathbf{b}} = 2iK'$, $A_\phi^{1/2} = k$.

Let us seek a function $b_0(x)$ that solves $\tilde{b}' = -2(A_\phi - \psi_0^2) + 4\psi_0'$ and is consistent with $B_\phi(t)$ for (6.3). Put $u = (x - x_0)/2$. Then this becomes

$$(b_0)_u = 4(\psi_0)_u + 4(\psi_0^2 - A_\phi) = 4(\psi_0)_u + 4A_\phi(\mathrm{sn}^2 u - 1). \tag{7.8}$$

Comparison of double poles of doubly periodic functions yields

$$(\psi_0)_u + A_\phi(\mathrm{sn}^2 u - 1) + \frac{2}{\Omega_{\mathbf{a}}}\frac{d}{du}\Big(\frac{\vartheta'}{\vartheta}\Big(\frac{u}{\Omega_{\mathbf{a}}}, \tau_0\Big)\Big) \equiv c_0 \in \mathbb{C}$$

with $\tau_0 = \Omega_{\mathbf{b}}/\Omega_{\mathbf{a}}$. Integrating this with (7.8) along $[0, u]$ and putting $u = 2K = \Omega_{\mathbf{a}}/2$, we have

$$b_0(x) = b_0(x_0) - \frac{2\mathcal{E}_{\mathbf{a}}}{\Omega_{\mathbf{a}}}(x - x_0) - \frac{8}{\Omega_{\mathbf{a}}}\frac{\vartheta'}{\vartheta}\Big(\frac{1}{2\Omega_{\mathbf{a}}}(x - x_0), \tau_0\Big),$$

since $2c_0 = -2\mathcal{E}_{\mathbf{a}}/\Omega_{\mathbf{a}}$ follows from

$$A_\phi\int_0^K (\mathrm{sn}^2 u - 1)du = -\int_0^{A_\phi^{1/2}} \sqrt{\frac{A_\phi - z^2}{1 - z^2}}dz = -\frac{\mathcal{E}_{\mathbf{a}}}{4} \qquad (z = A_\phi^{1/2}\mathrm{sn}\, u).$$

This is consistent with Corollary 6.1 if

$$b_0(x_0) = \beta_0 - \frac{2\mathcal{E}_{\mathbf{a}}}{\Omega_{\mathbf{a}}}x_0 = -\frac{8}{\Omega_{\mathbf{a}}}\big(\log(m_{21}^0 m_{12}^1) + \pi i(\theta_\infty + 1)\big) - \frac{2\mathcal{E}_{\mathbf{a}}}{\Omega_{\mathbf{a}}}x_0. \tag{7.9}$$

Therefore $b_0(x)$ satisfies

$$b_0'(x) = 2(\psi_0(x)^2 - A_\phi) + 4\psi_0'(x) \tag{7.10}$$

and $b_0(e^{i\phi}t) - e^{i\phi}B_\phi(t) \ll t^{-\delta}$ in $S(\phi, t_\infty, \kappa_0, \delta_0)$.

8. Modulus A_ϕ and the Boutroux equations

We examine a solution $A \in \mathbb{C}$ of the Boutroux equations. Let the branch of $A^{1/2}$ $(\neq 0)$ be fixed in such a way that $\mathrm{Re}\, A^{1/2} \ge 0,$ and $\mathrm{Im}\, A^{1/2} > 0$ if $\mathrm{Re}\, A^{1/2} = 0$. In accordance with [31, Appendix I] set

$$I_{\mathbf{a}}(A) = \int_{\mathbf{a}} \sqrt{\frac{A - z^2}{1 - z^2}} dz, \quad I_{\mathbf{b}}(A) = \int_{\mathbf{b}} \sqrt{\frac{A - z^2}{1 - z^2}} dz, \quad \mathcal{I}(A) = \frac{I_{\mathbf{a}}(A)}{I_{\mathbf{b}}(A)},$$

in which the cycles $\mathbf{a}$ and $\mathbf{b}$ are as in Figure 2.2 with $A_\phi = A$.

Lemma 8.1. *Let $A \in \mathbb{C}$. Then $\mathcal{I}(A) \in \mathbb{R}$ if and only if, for some $\phi \in \mathbb{R}$, A solves the Boutroux equations* $(\mathrm{BE})_\phi : \mathrm{Re}\, e^{i\phi} I_{\mathbf{a}}(A) = \mathrm{Re}\, e^{i\phi} I_{\mathbf{b}}(A) = 0.$

Proof. Suppose that $\mathcal{I}(A) = \rho \in \mathbb{R}$, and write $I_{\mathbf{a}}(A) = u + iv$, $I_{\mathbf{b}}(A) = U + iV$ with u, v, U, $V \in \mathbb{R}$. Then $u = \rho U$, $v = \rho V$, and hence $u/v = U/V = \tan\phi$ for some $\phi \in \mathbb{R}$, which implies $\mathrm{Re}\, e^{i\phi} I_{\mathbf{a}}(A) = \mathrm{Re}\, e^{i\phi} I_{\mathbf{b}}(A) = 0$. □

By Lemma 5.8,

$$\mathcal{I}'(A) = \frac{1}{2 I_{\mathbf{b}}(A)^2} (\omega_{\mathbf{a}}(A) I_{\mathbf{b}}(A) - \omega_{\mathbf{b}}(A) I_{\mathbf{a}}(A)) = -\frac{2\pi i}{I_{\mathbf{b}}(A)^2},$$
$$\omega_{\mathbf{a},\,\mathbf{b}}(A) = \int_{\mathbf{a},\,\mathbf{b}} \frac{dz}{\sqrt{(A - z^2)(1 - z^2)}},$$

which implies the following.

Lemma 8.2. *The map $\mathcal{I}(A)$ is conformal on $\mathbb{C}$ as long as $I_{\mathbf{b}}(A) \neq 0, \infty$.*

Near $A = \infty$, observing that

$$I_{\mathbf{a}}(A) = 4A^{1/2} \int_0^1 \sqrt{\frac{1 - z^2/A}{1 - z^2}} dz = 2\pi A^{1/2} (1 + O(A^{-1})),$$

and that

$$\begin{aligned} I_{\mathbf{b}}(A) =& 2A^{1/2} \int_1^{A^{1/2}} \sqrt{\frac{1 - z^2/A}{1 - z^2}} dz \\ =& -2iA^{1/2} \int_1^{A^{1/2}} \Big(\frac{1}{\sqrt{z^2 - 1}} + \frac{-z^2/A}{\sqrt{z^2 - 1}(1 + \sqrt{1 - z^2/A})} \Big) dz \\ =& -iA^{1/2} \log A \; (1 + O(|\log A|^{-1})), \end{aligned}$$

we have $\mathrm{Im}\,(1/\mathcal{I}(A)) = -(2\pi)^{-1} \log |A| \; (1 + o(1))$ as $A \to \infty$.

Lemma 8.3. *The set*

$$\mathcal{R}(\mathrm{BE}) = \mathcal{I}^{-1}(\mathbb{R}) = \{ A \in \mathbb{C} ;\; A \text{ solves } (\mathrm{BE})_\phi \text{ for some } \phi \in \mathbb{R} \}$$

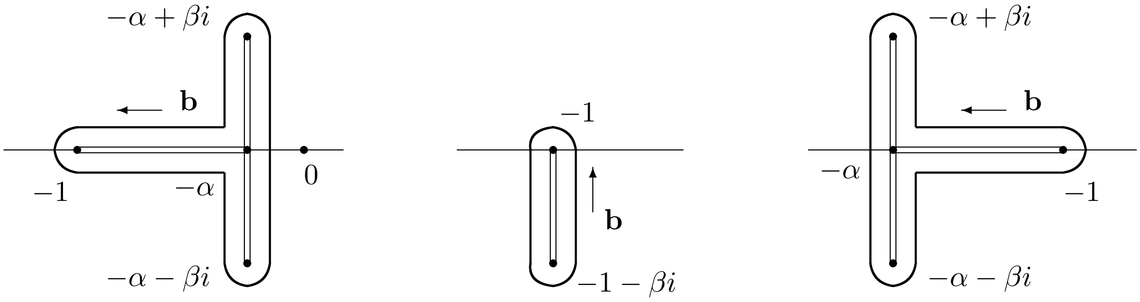
is bounded.

Let us observe the dependence of $A \in \mathcal{R}(\mathrm{BE})$ on ϕ or $t = \tan\phi$.

Since $I_{\mathbf{a}}(0) = 0$ and $I_{\mathbf{b}}(0) = 2i$, $A = 0$ solves $(\mathrm{BE})_{\phi=0}$. Conversely we may give the uniqueness lemma, which is crucial in discussing $(\mathrm{BE})_{\phi}$. This is proved by an argument similar to that in [27, §7].

Lemma 8.4. *If A solves* $(\mathrm{BE})_{\phi=0}$, *then* $A = 0$.

Proof. Suppose that $\mathrm{Re}\, I_{\mathbf{a}}(A) = \mathrm{Re}\, I_{\mathbf{b}}(A) = 0$. Then $I_{\mathbf{b}}(A)$ is pure imaginary, and $I_{\mathbf{b}}(A) = -\overline{I_{\mathbf{b}}(A)} = -I_{\overline{\mathbf{b}}}(\overline{A}) = I_{\mathbf{b}}(\overline{A})$, that is, $I_{\mathbf{b}}(A) - I_{\mathbf{b}}(\overline{A}) = 0$.



(a) $0 \le \mathrm{Re}\, A^{1/2} < 1$ (b) $\mathrm{Re}\, A^{1/2} = 1$ (c) $\mathrm{Re}\, A^{1/2} > 1$

FIGURE 8.1. Cycle **b**

(a) *Case where* $0 \le \mathrm{Re}\, A^{1/2} < 1$: Write $A^{1/2} = \alpha + i\beta$ with $0 \le \alpha < 1$, say, $\beta \ge 0$. Then the cycle **b** may be deformed in such a way that **b** surrounds anticlockwise the cuts $[-\alpha - i\beta, -\alpha + i\beta] \cup [-1, -\alpha]$, where $-\alpha - i\beta = -A^{1/2}$, $-\alpha + i\beta = -\overline{A}^{1/2}$ (Figure 8.1, (a)). The function $\sqrt{(A - z^2)(1 - z^2)}$ (respectively, $\sqrt{(\overline{A} - z^2)(1 - z^2)}$) may be treated on the plane with the cuts $[-1, -\alpha] \cup [-\alpha, -\alpha - i\beta]$ (respectively, $[-1, -\alpha] \cup [-\alpha, -\alpha + i\beta]$). We have

$$I_{\mathbf{b}}(A) - I_{\mathbf{b}}(\overline{A}) = \int_{\mathbf{b}} \Big(\sqrt{\frac{A - z^2}{1 - z^2}} - \sqrt{\frac{\overline{A} - z^2}{1 - z^2}} \Big) dz = (A - \overline{A}) I_{\mathbf{b}}(A, \overline{A}) = 0,$$

where

$$I_{\mathbf{b}}(A, \overline{A}) = \int_{\mathbf{b}} \frac{dz}{\sqrt{1 - z^2}(\sqrt{A - z^2} + \sqrt{\overline{A} - z^2})}.$$

To show $A \in \mathbb{R}$, suppose the contrary $A - \overline{A} \ne 0$. Dividing **b** into five parts, we have

$$I_{\mathbf{b}}(A, \overline{A}) = I_0^{\beta} + \tilde{I}_{\beta}^{0} + H_{-\alpha}^{-1} + \tilde{J}_0^{-\beta} + J_{-\beta}^{0},$$

in which

$$I_0^{\beta} = \int_0^{\beta} \frac{idt}{\sqrt{1 - (-\alpha + it)^2}(\sqrt{A - (-\alpha + it)^2} + \sqrt{\overline{A} - (-\alpha + it)^2})},$$

$$\tilde{I}_{\beta}^{0} = \int_{\beta}^{0} \frac{idt}{\sqrt{1 - (-\alpha + it)^2}(\sqrt{A - (-\alpha + it)^2} - \sqrt{\overline{A} - (-\alpha + it)^2})},$$

$$H_{-\alpha}^{-1} = \int_{-\alpha}^{-1} \frac{2dt}{\sqrt{1-t^2}(\sqrt{A-t^2} - \sqrt{\overline{A}-t^2})},$$
$$\tilde{J}_0^{-\beta} = \int_0^{-\beta} \frac{idt}{-\sqrt{1-(-\alpha+it)^2}(\sqrt{A-(-\alpha+it)^2} - \sqrt{\overline{A}-(-\alpha+it)^2})},$$
$$J_{-\beta}^0 = \int_{-\beta}^0 \frac{idt}{-\sqrt{1-(-\alpha+it)^2}(-\sqrt{A-(-\alpha+it)^2} - \sqrt{\overline{A}-(-\alpha+it)^2})}.$$

Then

$$(I_0^\beta + \tilde{I}_\beta^0) + (\tilde{J}_0^{-\beta} + J_{-\beta}^0) = \frac{2i}{A-\overline{A}} \int_0^\beta \left(\sqrt{\frac{A-(\alpha+it)^2}{1-(\alpha+it)^2}} - \overline{\sqrt{\frac{A-(\alpha+it)^2}{1-(\alpha+it)^2}}} \right) dt \in i\mathbb{R}$$

(for the branch of $\sqrt{(A-z^2)/(1-z^2)}$ see Section 5). The remaining integral $H_{-\alpha}^{-1}$ is

$$\begin{aligned} -\frac{1}{2} H_{-\alpha}^{-1} &= \frac{i}{A-\overline{A}} \int_\alpha^1 \frac{\sqrt{t^2-(\alpha+i\beta)^2} + \sqrt{t^2-(\alpha-i\beta)^2}}{\sqrt{1-t^2}} dt \\ &= \frac{2i}{A-\overline{A}} \int_\alpha^1 \frac{\operatorname{Re}\sqrt{t^2-\alpha^2+\beta^2-2i\alpha\beta}}{\sqrt{1-t^2}} dt \\ &= \frac{\sqrt{2}i}{A-\overline{A}} \int_\alpha^1 \frac{\sqrt{t^2-\alpha^2+\beta^2+\sqrt{(t^2-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}}}{\sqrt{1-t^2}} dt \in \mathbb{R}\setminus\{0\}. \end{aligned}$$

Hence $I_{\mathbf{b}}(A, \overline{A}) \neq 0$, yielding the contradiction $A - \overline{A} = 0$. In this case we have $A \in \mathbb{R}$.

(b) *Case where* $\operatorname{Re} A^{1/2} = 1$: Write $A^{1/2} = 1 + i\beta$, say $\beta \geq 0$ (cf. Figure 8.1, (b)). Then

$$\begin{aligned} I_{\mathbf{b}}(A) = \int_{\mathbf{b}} \sqrt{\frac{A-z^2}{1-z^2}} dz &= 2i \int_{-\beta}^0 \sqrt{\frac{(-1-i\beta)^2-(-1+it)^2}{1-(-1+it)^2}}\, dt \\ =2i \int_0^\beta \sqrt{\frac{t^2-\beta^2-2(t-\beta)i}{t^2-2ti}}\, dt &= -2\int_0^\beta \sqrt{\frac{\beta-t}{t(4+t^2)}} \sqrt{t^2+\beta t+4+2\beta i}\, dt \end{aligned}$$

with

$$\operatorname{Re}\sqrt{t^2+\beta t+4+2\beta i} = \sqrt{t^2+\beta t+4+\sqrt{(t^2+\beta t+4)^2+4\beta^2}} \geq 2\sqrt{2},$$

which implies $\operatorname{Re} I_{\mathbf{b}}(A) \neq 0$.

(c) *Case where* $\operatorname{Re} A^{1/2} > 1$: It is shown that $\operatorname{Re} I_{\mathbf{b}}(A) = 0$ implies $A \in \mathbb{R}$ by an argument similar to that in the case (a).

Thus in every case we have shown $A \in \mathbb{R}$ or $\operatorname{Re} I_{\mathbf{b}}(A) \neq 0$. We may examine $I_{\mathbf{a}}(A)$ and $I_{\mathbf{b}}(A)$ for each $A \in \mathbb{R}$ to conclude that $\operatorname{Re} I_{\mathbf{a}}(A) = \operatorname{Re} I_{\mathbf{b}}(A) = 0$ if and only if $A = 0$. This completes the proof. □

Corollary 8.5. *For every* $A \in \mathbb{C}$, $(I_{\mathbf{a}}(A), I_{\mathbf{b}}(A)) \neq (0, 0)$.

Corollary 8.6. *If* $\operatorname{Re} I_{\mathbf{b}}(A) = 0$, *then* $A = 0$.

Since $I_{\mathbf{a}}(1) = 4$, $I_{\mathbf{b}}(1) = 0$, the number $A = 1$ solves $(\mathrm{BE})_{\phi=\pm\pi/2}$. Observe that $\mathrm{Re}\, iI_{\mathbf{b}}(A) = 0$ implies $I_{\mathbf{b}}(A) = -I_{\mathbf{b}}(\overline{A})$. Then, similarly we have the following.

Lemma 8.7. *If A solves $(\mathrm{BE})_{\phi=\pm\pi/2}$, then $A = 1$.*

Corollary 8.8. *If $\mathrm{Re}\, iI_{\mathbf{b}}(A) = 0$, then $A = 1$.*

Lemma 8.9. *If $|\phi|$ is sufficiently small, equations $(\mathrm{BE})_{\phi}$ admit a solution $A_{\phi} = x(\phi) + iy(\phi)$ such that*

$$x(\phi) = -\frac{4\phi^2}{\log\phi}(1 + o(1)), \quad y(\phi) = -\frac{4\phi}{\log\phi}(1 + o(1)),$$

which is unique around $A = 0$.

Proof. Suppose that $|A|$ is small and $\mathrm{Re}\, A^{1/2} \geq 0$. Then

$$I_{\mathbf{a}}(A) = \int_{\mathbf{a}} \sqrt{\frac{A - z^2}{1 - z^2}}dz = 2\int_{-A^{1/2}}^{A^{1/2}} \sqrt{\frac{A - z^2}{1 - z^2}}dz = 2A\int_{-1}^{1} \sqrt{\frac{1 - t^2}{1 - At^2}}dt = \pi A + O(A^2),$$

$$\begin{aligned}
I_{\mathbf{b}}(A) &= \int_{\mathbf{b}} \sqrt{\frac{A - z^2}{1 - z^2}}dz = 2i\int_{A^{1/2}}^{1} \sqrt{\frac{z^2 - A}{1 - z^2}}dz \\
&= 2i\left(\int_{A^{1/2}}^{1} \frac{zdz}{\sqrt{1 - z^2}} - A\int_{A^{1/2}}^{1} \frac{dz}{\sqrt{1 - z^2}(z + \sqrt{z^2 - A})}\right) \\
&= \frac{i}{2}(4 + A\log A + O(A)).
\end{aligned}$$

From $\mathrm{Re}\, e^{i\phi}I_{\mathbf{a}}(A_{\phi}) = \mathrm{Re}\, e^{i\phi}I_{\mathbf{b}}(A_{\phi}) = 0$, that is,

$$\mathrm{Re}\,((A_{\phi} + O(A_{\phi}^2))(\cos\phi + i\sin\phi)) = \mathrm{Re}\,(i(4 + A_{\phi}\log A_{\phi} + O(A_{\phi}))(\cos\phi + i\sin\phi)) = 0$$

with $A_{\phi} = x(\phi) + iy(\phi)$, the conclusion follows. □

Similarly we have the following.

Lemma 8.10. *If $|\phi \mp \pi/2|$ is sufficiently small, equations $(\mathrm{BE})_{\phi}$ admit a solution $A_{\phi} = x(\phi) + iy(\phi)$ such that*

$$x(\phi) = 1 + \frac{4\tilde{\phi}_{\pm}^2}{\log\tilde{\phi}_{\pm}}(1 + o(1)), \quad y(\phi) = \frac{4\tilde{\phi}_{\pm}}{\log\tilde{\phi}_{\pm}}(1 + o(1))$$

with $\phi = \pm\pi/2 + \tilde{\phi}_{\pm}$.

Lemma 8.11. *Suppose that $0 < |\phi_0| < \pi/2$ and that $A(\phi_0)$ solves $(\mathrm{BE})_{\phi=\phi_0}$. Then there exists a curve $\Gamma(\phi_0)$ given by $A = A(\phi_0, \phi)$ for $|\phi| \leq \pi/2$, where $A(\phi_0, \phi)$ has the properties :*

(i) *$A(\phi_0, \phi_0) = A(\phi_0)$, $A(\phi_0, 0) = 0$, $A(\phi_0, \pm\pi/2) = 1$;*

(ii) *$A(\phi_0, \phi)$ is continuous in ϕ for $|\phi| \leq \pi/2$ and smooth for $0 < |\phi| < \pi/2$;*

(iii) *$A(\phi_0, \phi)$ solves $(\mathrm{BE})_{\phi}$ for $|\phi| \leq \pi/2$.*

Proof. Set

$$A = x + iy, \quad I_{\mathbf{a}}(A) = u(A) + iv(A), \quad I_{\mathbf{b}}(A) = U(A) + iV(A)$$

with x, y, $u(A)$, $v(A)$, $U(A)$, $V(A) \in \mathbb{R}$. Then A solves $(\mathrm{BE})_\phi$ if and only if

$$\mathrm{Re}\, e^{i\phi} I_{\mathbf{a}}(A) = u(A)\cos\phi - v(A)\sin\phi = \mathrm{Re}\, e^{i\phi} I_{\mathbf{b}}(A) = U(A)\cos\phi - V(A)\sin\phi = 0,$$

that is,

$$u(A) - v(A)t = U(A) - V(A)t = 0 \quad \text{with } t = \tan\phi. \tag{8.1}$$

By the Cauchy-Riemann relations the Jacobian for (8.1) is

$$\begin{aligned} J(v,V,t;A) &= \det \begin{pmatrix} v_y - tv_x & -v_x - tv_y \\ V_y - tV_x & -V_x - tV_y \end{pmatrix} = (1+t^2)(v_xV_y - v_yV_x) \\ &= -\frac{i}{8}(1+t^2)(\omega_{\mathbf{a}}(A)\overline{\omega_{\mathbf{b}}(A)} - \overline{\omega_{\mathbf{a}}(A)}\omega_{\mathbf{b}}(A)) = -\frac{1}{4}(1+t^2)\mathrm{Im}\,(\overline{\omega_{\mathbf{a}}(A)}\omega_{\mathbf{b}}(A)) \\ &= -\frac{1}{4}(1+t^2)|\omega_{\mathbf{a}}(A)|^2 \mathrm{Im}\,\frac{\omega_{\mathbf{b}}(A)}{\omega_{\mathbf{a}}(A)} < 0, \ \neq \infty, \end{aligned} \tag{8.2}$$

provided that $A \neq 0, 1$. By supposition, since $A(\phi_0) \neq 0, 1$, there exists a function $A_0(\phi)$ with the properties:

(a) $A_0(\phi_0) = A(\phi_0)$;

(b) $A_0(\phi)$ is smooth for $|\phi - \phi_0| < \varepsilon_*$, ε_* being sufficiently small;

(c) $A_0(\phi)$ solves (8.1), i.e. $(\mathrm{BE})_\phi$ for $|\phi - \phi_0| < \varepsilon_*$ and is a unique solution in a small neighbourhood of $A(\phi_0)$.

Let us consider the case $0 < \phi_0 < \pi/2$. Denote by $\mathcal{F}(\phi_0)$ the family of functions $\hat{A}_\nu(\phi)$ with the properties:

(a_ν) $\hat{A}_\nu(\phi_0) = A(\phi_0)$;

(b_ν) $\hat{A}_\nu(\phi)$ is smooth for $\phi_0 - \varepsilon_* < \phi < \phi_\nu < \pi/2$;

(c_ν) $\hat{A}_\nu(\phi)$ solves (8.1) for $\phi_0 - \varepsilon_* < \phi < \phi_\nu$.

Then $A_0(\phi) \in \mathcal{F}(\phi_0)$. For any $\hat{A}_\nu(\phi)$, $\hat{A}_{\nu'}(\phi) \in \mathcal{F}(\phi_0)$ with $\phi_\nu > \phi_{\nu'}$, $\hat{A}_\nu(\phi) \equiv \hat{A}_{\nu'}(\phi)$ holds if $\phi_0 - \varepsilon_* < \phi < \phi_{\nu'}$. Let $\hat{A}_\infty(\phi)$ be the maximal extension of all $\hat{A}_\nu(\phi) \in \mathcal{F}(\phi_0)$, and set $\phi_\infty = \sup_\nu \phi_\nu$. Then $\hat{A}_\infty(\phi)$ solves (8.1) and is smooth for $\phi_0 - \varepsilon_* < \phi < \phi_\infty$. Suppose that $\phi_\infty < \pi/2$. Since $\hat{A}_\infty(\phi_\infty) \neq 0, 1$, the Jacobian $J(v, V, \tan\phi_\infty; \hat{A}_\infty(\phi_\infty))$ does not vanish, which implies that $\hat{A}_\infty(\phi)$ may be extended beyond ϕ_∞. This is a contradiction. Hence we have $\phi_\infty = \pi/2$, and by Lemma 8.7, $\hat{A}_\infty(\pi/2) = 1$. Similarly we may construct a lower extension $\hat{A}^*_\infty(\phi)$ for $0 \le \phi < \phi_0 + \varepsilon_*$ satisfying $\hat{A}^*_\infty(0) = 0$, and then have the extension $A^+(\phi_0, \phi)$ for $0 \le \phi \le \pi/2$. The case $-\pi/2 < \phi_0 < 0$ may be treated in the same way to obtain $A^-(\phi_0, \phi)$ for $-\pi/2 \le \phi \le 0$. For a given ϕ_0 satisfying, say, $0 < \phi_0 < \pi/2$, combining $A^+(\phi_0, \phi)$ with $A^-(\phi_-, \phi)$, where $\phi_- < 0$ close to 0 is such that A_{ϕ_-} is a solution given in Lemma 8.9, we obtain the desired extension $A(\phi_0, \phi)$ for $|\phi| \le \pi/2$. Thus the lemma is proved. □

Corollary 8.12. *Under the same supposition as in Lemma* 8.11*,* $(d/d\phi)A(\phi_0,\phi)\neq 0$ *for* $0<|\phi|<\pi/2$*. Furthermore,* $(d/d\phi)\mathcal{I}(A(\phi_0,\phi))>0$ *or* <0 *for* $0<\phi<\pi/2$*, and so for* $-\pi/2<\phi<0$*.*

Proof. From (8.1) it follows that

$$J(v,V,\tan\phi;A(\phi_0,\phi))\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix}-\begin{pmatrix}v(A(\phi_0,\phi))\\ V(A(\phi_0,\phi))\end{pmatrix}\equiv \mathbf{o},$$

where $A(\phi_0,\phi)=x(t)+iy(t)$. By Corollary 8.5 and (8.2), $(x'(t),y'(t))\neq(0,0)$ if $0<|\phi|<\pi/2$, i.e. $t\in\mathbb{R}\setminus\{0\}$, and then $(d/d\phi)A(\phi_0,\phi)=(x'(t)+iy'(t))/\cos^2\phi\neq 0$. Since $\mathcal{I}(A(\phi_0,\phi))\in\mathbb{R}$ by Lemma 8.1, we have

$$\frac{d}{d\phi}\mathcal{I}(A(\phi_0,\phi))=\frac{d}{d\phi}A(\phi_0,\phi)\frac{-2\pi i}{I_{\mathbf{b}}(A(\phi_0,\phi))^2}\in\mathbb{R}\setminus\{0\}$$

for $0<|\phi|<\pi/2$, from which the conclusion follows. □

Proposition 8.13. *For each* ϕ_* *such that* $|\phi_*|\leq\pi/2$*, equations* $(\mathrm{BE})_{\phi=\phi_*}$ *admit a unique solution* $A_{\phi_*}\in\mathbb{C}$*.*

Proof. Let $\hat{\phi}_0$ be so close to 0 that $A_{\hat{\phi}_0}$ is a solution given in Lemma 8.9. Lemma 8.11 with $\phi_0=\hat{\phi}_0$ provides a curve $\Gamma(\hat{\phi}_0)$ containing a solution of $(\mathrm{BE})_{\phi=\phi_*}$ for each fixed ϕ_*. It remains to show the uniqueness of a solution for $\phi_*\neq 0,\pm\pi/2$. Suppose that A_{ϕ_*} and A'_{ϕ_*} solve $(\mathrm{BE})_{\phi=\phi_*}$. Then, by Lemmas 8.4 and 8.11, there exist curves $\Gamma(\phi_*)$ and $\Gamma'(\phi_*)$ such that $\Gamma(\phi_*)\ni 0,A_*$, $\Gamma'(\phi_*)\ni 0,A'_*$. Then, by (8.2) (or the conformality of Lemma 8.2), we have $\Gamma(\phi_*)=\Gamma'(\phi_*)\ni A_{\phi_*}=A'_{\phi_*}$, which completes the proof. □

By the uniqueness above we easily have the following.

Lemma 8.14. *For* $\phi\in\mathbb{R}$*,* $(\mathrm{BE})_\phi$ *admit a unique solution* A_ϕ*, which satisfies*

$$A_{\phi\pm\pi}=A_\phi,\quad A_{-\phi}=\overline{A_\phi}.$$

Lemma 8.15. *Each* A_ϕ *given in Lemma* 8.14 *satisfies* $0\leq\operatorname{Re}A_\phi\leq 1$*. For* $0<\phi<\pi/2$ *(respectively,* $-\pi/2<\phi<0$*),* $(d/d\phi)\operatorname{Re}A_\phi>0$ *(respectively,* <0*).*

Proof. Let $A_\phi=x(t)+iy(t)$, $t=\tan\phi$. Then, by Corollary 8.12,

$$(d/dt)\mathcal{I}(A_\phi)=(x'(t)+iy'(t))(-2\pi i)I_{\mathbf{b}}(A_\phi)^{-2}\in\mathbb{R}\setminus\{0\}$$

for $0<|\phi|<\pi/2$. This yields $x'(t)(U_*^2-V_*^2)-2y'(t)U_*V_*=0$, where $I_{\mathbf{b}}(A_\phi)^{-1}=U_*+iV_*$. Suppose that, $x'(t_0)=0$ and $0<\operatorname{Re}A_{\phi_0}<1$, for some $t_0=\tan\phi_0\neq 0,\pm\infty$. Since $y'(t_0)\neq 0$, $U_*V_*=0$. If $U_*=0$, then $\operatorname{Re}I_{\mathbf{b}}(A_{\phi_0})=0$, and hence $A_{\phi_0}=0$, i.e. $\phi_0=0$ by Corollary 8.6. If $V_*=0$, then $\operatorname{Re}iI_{\mathbf{b}}(A_{\phi_0})=0$, and hence $A_{\phi_0}=1$, i.e. $\phi_0=\pm\pi/2$ by Corollary 8.8. Thus we have shown that $x'(t)>0$ or $x'(t)<0$ for $0<|\phi|<\pi/2$, $t=\tan\phi$, which implies $0\leq\operatorname{Re}A_\phi\leq 1$. □

Remark 8.1. In the proof above, it is easy to see that $y'(t)=0$ occurs if and only if $U_*=\pm V_*$, that is, $\phi=\pm\pi/4$, $\pm 3\pi/4$.

Lemmas 8.9, 8.11, 8.15, Proposition 8.13 and Remark 8.1 leads to the following.

Proposition 8.16. *There exists a Jordan closed curve* $\Gamma_0 = \{A_\phi\, ;\ |\phi| \le \pi/2\}$ *with the properties*:

(i) $A_0 = 0$, $A_{\pm\pi/2} = 1$;

(ii) A_ϕ *is smooth for* $0 < |\phi| < \pi/2$;

(iii) *for every* $\phi, |\phi| \le \pi/2$, A_ϕ *solves* $(\mathrm{BE})_\phi$.

By the properties above the trajectory of A_ϕ for $|\phi| \le \pi/2$ is as in Figure 8.2, (a).

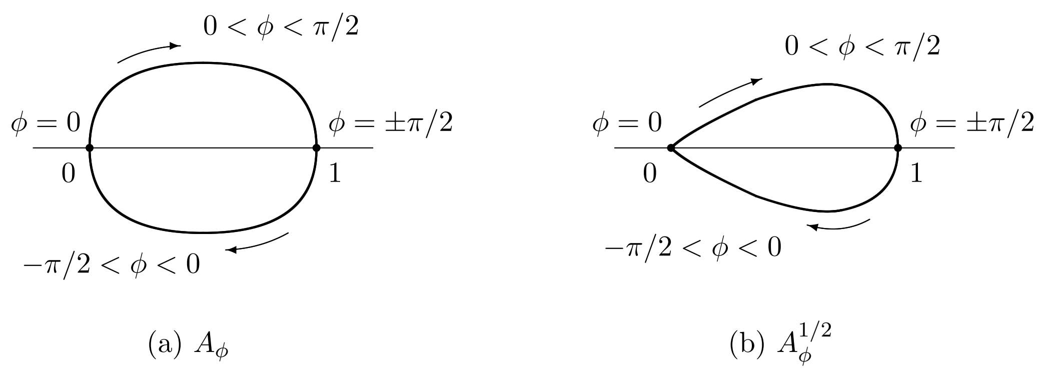


FIGURE 8.2. Rough drawings of the trajectories A_ϕ and $A_\phi^{1/2}$

The properties of A_ϕ are summarised as follows.

Proposition 8.17. (1) *For* $|\phi| \le \pi/2$, *the Boutroux equations* $(\mathrm{BE})_\phi$ *have a unique solution* A_ϕ *with the properties*:

(i) $A_0 = 0$, $A_{\pm\pi/2} = 1$;

(ii) A_ϕ *is smooth in* ϕ *such that* $0 < |\phi| < \pi/2$;

(iii) *for* $0 < \phi < \pi/2$ (*respectively,* $-\pi/2 < \phi < 0$), $x(t) = \mathrm{Re}\, A_\phi$, $t = \tan\phi$ *satisfies* $x'(t) > 0$ (*respectively,* $x'(t) < 0$), *and* $y(t) = \mathrm{Im}\, A_\phi$ *satisfies* $y'(t) = 0$ *if and only if* $\phi = \pm\pi/4, \pm 3\pi/4$;

(iv) $0 < \mathrm{Re}\, A_\phi < 1$ *for* $0 < |\phi| < \pi/2$, *and* $\mathrm{Im}\, A_\phi > 0$ *for* $0 < \phi < \pi/2$ *and* < 0 *for* $-\pi/2 < \phi < 0$.

(2) *For* $\phi \in \mathbb{R}$, A_ϕ *may be extended by using the relations* $A_{-\phi} = \overline{A_\phi}$, $A_{\phi\pm\pi} = A_\phi$.

Remark 8.2. By Lemma 8.9, around $\phi = 0$, $A_\phi = x(\phi) + iy(\phi)$ is expressed as $A_\phi = \pm i 2^{3/2}(xL(x)^{-1})^{1/2}(1+o(1))$ and $A_\phi^{1/2} = \pm e^{\pi i/4} 2^{3/4}(xL(x)^{-1})^{1/4}(1+o(1))$ as $x = x(\phi) = 4\phi^2 L(\phi)^{-1}(1+o(1)) \to 0$, where $L(x) = |\log x|$. By Lemma 8.10, around $\phi = \pm\pi/2$, $A_\phi = 1 - \frac{1}{4}y^2 L(y)(1+o(1)) + iy$ and $A_\phi^{1/2} = 1 - \frac{1}{8}y^2 L(y)(1+o(1)) + iy/2$ as $y = y(\phi) = -4\tilde{\phi} L(\tilde{\phi})^{-1}(1+o(1)) \to 0$ ($\tilde{\phi} = \phi \mp \pi/2$). Taking these local shapes around $\phi = 0$, and $\pm\pi/2$ into account, which are important in finding the Stokes graph, we have a rough drawing of the trajectory $A_\phi^{1/2}$ as in Figure 8.2, (b).

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