On a general singular solution of the fifth Painlevé equation along the positive real axis

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To the memory of Professor Walter Hayman

Abstract

We propose a system of nonlinear equations equivalent to the fifth Painlevé equation, which enables us to examine the general singular solution given by Andreev and Kitaev along the positive real axis. We present a two-parameter family of asymptotic solutions corresponding to this general singular solution, and pose a conjecture.

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1 Introduction

The fifth Painlevé equation

$$\frac{d^2y}{dt^2} = \Big(\frac{1}{2y} + \frac{1}{y-1}\Big)\Big(\frac{dy}{dt}\Big)^2 - \frac{1}{t}\frac{dy}{dt}$$

$$+ \frac{(y-1)^2}{8t^2}\Big((\theta_0 - \theta_1 + \theta_\infty)^2 y - \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{y}\Big) + (1 - \theta_0 - \theta_1)\frac{y}{t} - \frac{y(y+1)}{2(y-1)} \qquad (1.1)$$

with θ_0, θ_1, $\theta_\infty \in \mathbb{C}$, which defines nonlinear special functions meromorphic on the universal covering space of $\mathbb{C} \setminus \{0\}$, follows from the isomonodromy deformation of the two-dimensional linear system

$$\frac{d\Xi}{d\xi} = \Big(\frac{t}{2}\sigma_3 + \frac{\mathcal{A}_0}{\xi} + \frac{\mathcal{A}_1}{\xi - 1}\Big)\Xi, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (1.2)$$

$$\mathcal{A}_0 = \begin{pmatrix} z+\theta_0/2 & -u(z+\theta_0) \\ z/u & -z-\theta_0/2 \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} -z-(\theta_0+\theta_\infty)/2 & uy(z+(\theta_0-\theta_1+\theta_\infty)/2) \\ -(uy)^{-1}(z+(\theta_0+\theta_1+\theta_\infty)/2) & z+(\theta_0+\theta_\infty)/2 \end{pmatrix}.$$

The isomonodromy deformation of this system with respect to the parameter t is governed by the system of equations for $(y,z,u)=(y(t),z(t),u(t))$

$$t\frac{dy}{dt} = ty - 2z(y-1)^2 - (y-1)\Big(\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)y - \frac{1}{2}(3\theta_0+\theta_1+\theta_\infty)\Big), \tag{1.3}$$

$$t\frac{dz}{dt} = yz\Big(z+\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)\Big) - \frac{(z+\theta_0)}{y}\Big(z+\frac{1}{2}(\theta_0+\theta_1+\theta_\infty)\Big), \tag{1.4}$$

$$t\frac{d}{dt}\log u = -2z-\theta_0+y\Big(z+\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)\Big)+\frac{1}{y}\Big(z+\frac{1}{2}(\theta_0+\theta_1+\theta_\infty)\Big), \tag{1.5}$$

which is equivalent to (1.1) [5, Appendix C], [4]; that is, $y(t)$ solves (1.1) if and only if the monodromy data for (1.2) remain invariant under a small change of t. Using this fact combined with the WKB-analysis, for (1.1) Andreev and Kitaev [1] obtained families of solutions near $t=0$ and $t=\infty$ on the positive real axis, and connection formulas for these solutions. According to the associated monodromy data these solutions are classified and their asymptotic expressions are presented. Among them a general singular solution has a sequence of poles along the positive real axis [1, Theorem 4.1]:

Theorem 1.1. *Let* $\beta_0\in\mathbb{C}$, $1/4<\mathrm{Re}\,\beta_0<3/4$, $\hat{u}\in\mathbb{C}\setminus\{0\}$ *and* $\hat{v}\in\mathbb{C}\setminus\{0\}$. *There exists the unique solution of system* (1.3)-(1.5) *with asymptotic expression*

$$\begin{aligned} y &= \frac{\cos^2\tilde{x}}{\sin^2\tilde{x}} + O(t^{-1+2|\mathrm{Re}\,\beta_0-1/2|}), \\ z &= \frac{\cos\tilde{x}\sin\tilde{x}}{4(\cos\tilde{x}-\sin\tilde{x})^2}t + O(t^{2|\mathrm{Re}\,\beta_0-1/2|}), \\ u &= \hat{u}e^{t/2+\pi\mathrm{i}/2}\frac{\sin\tilde{x}}{\cos\tilde{x}}(1+o(1)) \end{aligned}$$

as $t\to\infty$ *on* $\mathbb{R}^+\cap(\bigcap_{k\in\mathbb{Z}}\{|t-t_k|>\varepsilon\})$, *where*

$$\tilde{x} = \frac{t}{4}+\Big(\frac{1}{4\mathrm{i}}-\frac{\beta_0}{2\mathrm{i}}\Big)\log t - \frac{1}{2\mathrm{i}}\log\Big(-\frac{\hat{v}}{\sqrt{2}}e^{\pi\mathrm{i}/4}\Big),$$

and t_k *is such that*

$$\frac{t_k}{4}+\Big(\frac{1}{4\mathrm{i}}-\frac{\beta_0}{2\mathrm{i}}\Big)\log t_k - \frac{1}{2\mathrm{i}}\log\Big(-\frac{\hat{v}}{\sqrt{2}}e^{\pi\mathrm{i}/4}\Big) = \frac{\pi}{4}k.$$

In the statement above, the integration constants β_0, $\hat{v}$, $\hat{u}$ are contained in the monodromy data. For the solution $y=y(t)$, the cluster set $\bigcap_{\tau>\tau_0}\{y(t)\in\mathbb{C};t>\tau\}$ is an unbounded continuum, and, as far as the author knows, it seems difficult to examine this solution by the linearlisation method like e.g. that in [11].

In this paper we propose another approach to this general singular solution. In the context of studying the elliptic asymptotic representation of a general solution of (1.1), the author found the system of equations [13, (6.8), (6.9)]

$$\begin{aligned} 4(\psi_t)^2 &= (1-\psi^2)(A_\phi - \psi^2) - (1-\phi^2)(4(\theta_0+\theta_1)\psi - b)t^{-1} \\ &\qquad + 4(2(\theta_0-\theta_1)\theta_\infty\psi + (\theta_0-\theta_1)^2 + \theta_\infty^2)t^{-2}, \\ b_t &= -2(A_\phi - \psi^2) + 4\psi' + (4(\theta_0+\theta_1)\psi - b)t^{-1} \end{aligned}$$

with

$$\psi = \frac{y+1}{y-1}$$

equivalent to (1.1), and used it in examining the error term of the elliptic expression. For each $\phi \in \mathbb{R}$, the quantity A_ϕ is a unique solution of the Boutroux equations

$$\operatorname{Re} e^{i\phi} \int_{\mathbf{a}} \sqrt{\frac{A_\phi - z^2}{1-z^2}}\,dz = \operatorname{Re} e^{i\phi} \int_{\mathbf{b}} \sqrt{\frac{A_\phi - z^2}{1-z^2}}\,dz = 0,$$

a, **b** denoting basic cycles on the elliptic curve $w^2 = (1-z^2)(A_\phi - z^2)$ [13, §§2 and 7]. Along the ray $\arg t = \phi$, $\phi \neq k\pi/2$ $(k \in \mathbb{Z})$ a general solution admits an asymptotic representation in terms of the Jacobi sn-function [14, Chap. 22] (for the elliptic representation of other Painlevé transcendents, see e.g. [3, 6–10]). If $\arg t = \phi = 0$, then $A_0 = 0$ is a unique solution of the Boutroux equations, and the system above is

$$\begin{aligned} 4(\psi_t)^2 &= -\psi^2(1-\psi^2) - (1-\phi^2)(4(\theta_0+\theta_1)\psi - b)t^{-1} \qquad (1.6) \\ &\qquad + 4(2(\theta_0-\theta_1)\theta_\infty\psi + (\theta_0-\theta_1)^2 + \theta_\infty^2)t^{-2}, \\ b_t &= 2\psi^2 + 4\psi' + (4(\theta_0+\theta_1)\psi - b)t^{-1} \qquad (1.7) \end{aligned}$$

along the positive real axis. We present a two-parameter family of asymptotic solutions of this system, and give an explicit expression of the error term of the general singular solution. Furthermore a conjecture on its asymptotic expansion is posed. In regarding the general singular solution on $\mathbb{R}^+$ as a degenerate case of the elliptic expression, it is quite natural to deal with (1.6) and (1.7) in examining such a solution, and results of this paper correspond to a degenerate version of the error term of the elliptic expression discussed in [13].

Throughout this paper, for complex-valued functions f and g, we write $f \ll g$ or $g \gg f$ if $|f| = O(|g|)$.

2 Results and a conjecture

For γ_0, $t_0 \in \mathbb{C}$, t_∞, R_0 and $\delta_0 > 0$, let

$$\Sigma(t_\infty, R_0, \gamma_0) = \{t\,;\ \operatorname{Re} t > t_\infty,\ |\operatorname{Im} t + 2(\operatorname{Im}\gamma_0)\log|t|\,| < R_0\},$$

$$\begin{aligned}&\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)=\Sigma(t_\infty,R_0,\gamma_0)\setminus\bigcup_{t_k\in Z_0}\{|t-t_k|<\delta_0\},\\ &Z_0=\{t_k\,;\ \ (t_k-t_0)/2+\gamma_0\log t_k=k\pi/2,\ k\in\mathbb{Z}\}.\end{aligned}$$

Remark 2.1. Since $\Sigma(t_\infty,R_0,\gamma_0)$ is given by

$$-\frac{R_0}{\log|t|}-2\mathrm{Im}\,\gamma_0<\frac{\mathrm{Im}\,t}{\log|t|}<\frac{R_0}{\log|t|}-2\mathrm{Im}\,\gamma_0,$$

this contains the positive real axis as $t\to\infty$ if and only if $\mathrm{Im}\,\gamma_0=0$, i.e. $\gamma_0\in\mathbb{R}$. If $\mathrm{Im}\,\gamma_0=0$, then $\Sigma(t_\infty,R_0,\gamma_0)$ is the strip $|\mathrm{Im}\,t|<R_0$, $\mathrm{Re}\,t>t_\infty$, and if $\mathrm{Im}\,\gamma_0\neq0$, then this is a region such that $|\mathrm{Im}\,t+2(\mathrm{Im}\,\gamma_0)\log|t||<R_0$. The sequence in Z_0 satisfies

$$t_k=\pi k-2\gamma_0\log(\pi k)+O(1),\quad \frac{\mathrm{Im}\,t_k}{\log|t_k|}=-2\mathrm{Im}\,\gamma_0+o(1),$$

and hence is located along the centre line of $\Sigma(t_\infty,R_0,\gamma_0)$.

The expression of the general singular solution $(y(t),z(t))$ in Theorem 1.1 is also valid in a domain of the type $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$, R_0 and small δ_0 being given numbers, and t_∞ sufficiently large [1, §9], [2, Remark 3.3].

Remark 2.2. In $\Sigma(t_\infty,R_0,\gamma_0)$, it is easy to see that $|e^{\mathrm{i}t/2}t^{\mathrm{i}\gamma_0}|^{\pm1}\ll1$, and hence $|\cos((t-t_0)/2+\gamma_0\log t)|\ll1$. In $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$ with $\mathrm{i}\gamma_0=1/2-\beta_0$ the error estimates $O(t^{-1+2|\mathrm{Re}\,\beta_0-1/2|})$ and $O(t^{2|\mathrm{Re}\,\beta_0-1/2|})$ in Theorem 1.1 may be replaced by $O(t^{-1})$ and $O(1)$, respectively.

Theorem 2.1. *Suppose that $\theta_0-\theta_1=\theta_\infty=0$. Then system* (1.6), (1.7) *possesses a two-parameter family of solutions $\{(\psi(t_0,\gamma_0;t),b(t_0,\gamma_0;t))\,;\ t_0,\gamma_0\in\mathbb{C}\}$ such that*

$$\begin{aligned}&\psi=\cos^{-1}((t-t_0)/2+\gamma_0\log t+g(t)),\\ &b/2-2\psi=-4\gamma_0+2\tan((t-t_0)/2+\gamma_0\log t+g(t))+\beta(t),\end{aligned}$$

where, for any integer N, $g(t)$ and $\beta(t)$ admit the asymptotic representations

$$\begin{aligned}&g(t)=\sum_{\nu=1}^{N}p_\nu(\cos\hat t,\sin\hat t)t^{-\nu}+O(t^{-N-1}),\\ &\beta(t)=\sum_{\nu=1}^{N}q_\nu(\cos\hat t,\sin\hat t)t^{-\nu}+O(t^{-N-1}),\quad \hat t=(t-t_0)/2+\gamma_0\log t,\end{aligned}$$

as $t\to\infty$ through $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$, where $p_\nu(\xi,\eta)$, $q_\nu(\xi,\eta)$ are polynomials in ξ, η of total degree $=2\nu$, δ_0 a given small number, R_0 a given number, and t_∞ a sufficiently large number depending on (δ_0,R_0).

Furthermore, for $\hat t$, $g(t)$ above

Corollary 2.2. *In* $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$*, the fifth Painlevé equation* (1.1) *admits a family of solutions* $\{y(t_0,\gamma_0;t)\,;\ t_0,\gamma_0\in\mathbb{C}\}$ *with*

$$y(t_0,\gamma_0;t)=\frac{2}{1-\cos(\hat{t}+g(t))}-1=\frac{\cos^2(\hat{t}/2+g(t)/2)}{\sin^2(\hat{t}/2+g(t)/2)}.$$

Example 2.1. Suppose that $\theta_0-\theta_1=\theta_\infty=0$, and let $a=2(2\theta_0-1)$. For $N=2$ we have

$$\begin{aligned}
p_1(\cos\hat{t},\sin\hat{t}) =&\frac{1}{2}\cos 2\hat{t}+a\sin\hat{t}+\frac{1}{8}(4\gamma_0^2+2a^2+1),\\
q_1(\cos\hat{t},\sin\hat{t}) =&-\frac{1}{2}(8\gamma_0^2+a^2+2),\\
p_2(\cos\hat{t},\sin\hat{t}) =&-\frac{\gamma_0}{4}\cos 4\hat{t}-\frac{1}{16}(4\gamma_0^2-3)\sin 4\hat{t}-\frac{7a}{12}\cos 3\hat{t}-\frac{a\gamma_0}{3}\sin 3\hat{t}\\
&-2\gamma_0\cos 2\hat{t}+\frac{1}{8}(12\gamma_0^2-2a^2+7)\sin 2\hat{t}+\frac{a}{8}(4\gamma_0^2+2a^2-29)\cos\hat{t}\\
&-5a\gamma_0\sin\hat{t}-\frac{\gamma_0}{16}(12\gamma_0^2+18a^2+5),\\
q_2(\cos\hat{t},\sin\hat{t}) =&4\gamma_0\cos 2\hat{t}+(4\gamma_0^2-1)\sin 2\hat{t}+4a\cos\hat{t}+8a\gamma_0\sin\hat{t}+\frac{\gamma_0}{2}(4\gamma_0^2+4a^2+3)
\end{aligned}$$

(see the proof of Theorem 2.1).

Let $\mathrm{i}\gamma_0=1/2-\beta_0$ and $\mathrm{i}t_0=\log(-\hat{v}e^{\pi\mathrm{i}/4}/\sqrt{2})$. Denote by $(y_*(t),z_*(t))$ the general singular solution given in Theorem 1.1. In $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$, we may suppose that $y_*(t)$ is expressed in the form

$$y_*(t)=\frac{\cos^2(\tilde{x}+g_*(t)/2)}{\sin^2(\tilde{x}+g_*(t)/2)}=\frac{1+\cos(2\tilde{x}+g_*(t))}{1-\cos(2\tilde{x}+g_*(t))},\quad g_*(t)\ll t^{-1}. \tag{2.1}$$

Let $(\psi_*,b_*)=(\psi_*(t),b_*(t))$ and $\beta_*(t)$ be such that

$$\begin{aligned}
&\psi_*=(y_*+1)/(y_*-1)=\cos^{-1}((t-t_0)/2+\gamma_0\log t+g_*(t)),\quad b_*=R(t,y_*,z_*),\\
&b_*/2-2\psi_*=2\mathrm{i}(1-2\beta_0)+2\tan((t-t_0)/2-\mathrm{i}(1/2-\beta_0)\log t+g_*(t))+\beta_*(t),
\end{aligned}$$

where $R(t,y,z)$ is a rational function in (t,y,z) resulting from the insertion of (1.3) into the right-hand side of (3.1) with $A_\phi=0$.

Theorem 2.3. *Suppose that* $\theta_0-\theta_1=\theta_\infty=0$*. Let* $\psi(t_0,\gamma_0;t)$*,* $b(t_0,\gamma_0;t)$*,* $y(t_0,\gamma_0;t)$ *and* $g(t)$ *be as in Theorem* 2.1 *and Corollary* 2.2*. Then* $\psi_*(t)\equiv\psi(t_0,\gamma_0;t)$*,* $b_*(t)\equiv b(t_0,\gamma_0;t)$*,* $y_*(t)\equiv y(t_0,\gamma_0;t)$ *and* $g_*(t)\equiv g(t)$ *in* $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$ *with* $\mathrm{i}\gamma_0=1/2-\beta_0$*,* $\mathrm{i}t_0=\log(-\hat{v}e^{i\pi/4}/\sqrt{2})$*.*

For any θ_0, θ_1, $\theta_\infty\in\mathbb{C}$ as well, $(g_*(t),\beta_*(t))$ corresponding to the general singular solution $(y_*(t),z_*(t))$ of Theorem 1.1 is defined in the same way as above. Then we have the asymptotic formula, at least, in t^{-1}, t^{-2}.

Theorem 2.4. *In the case of general parameters* θ_0, θ_1, $\theta_\infty \in \mathbb{C}$, $g_*(t)$ *and* $\beta_*(t)$ *admit the asymptotic representations*

$$\begin{aligned} g_*(t) =&(p_1^*(\cos\hat{t},\sin\hat{t})+a_2)t^{-1}+\Big(p_2^*(\cos\hat{t},\sin\hat{t})-a_2\sin 2\hat{t} \\ &+a_*a_2\cos\hat{t}-2a_1\sin\hat{t}-\frac{2a_1}{\sin\hat{t}}-\frac{2a_2\cos\hat{t}}{\sin\hat{t}}+c_2\Big)t^{-2}+O(t^{-3}), \\ \beta_*(t) =&(q_1^*(\cos\hat{t},\sin\hat{t})-2a_2)t^{-1} \\ &+\Big(q_2^*(\cos\hat{t},\sin\hat{t})+\frac{4a_1}{\sin\hat{t}}+\frac{4a_2\cos\hat{t}}{\sin\hat{t}}+c_2'\Big)t^{-2}+O(t^{-3}) \end{aligned}$$

as $t\to\infty$ *through* $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$. *Here*

(i) $a_*=2(\theta_0+\theta_1-1)$, $a_1=2(\theta_0-\theta_1)\theta_\infty$, $a_2=(\theta_0-\theta_1)^2+\theta_\infty^2$, $\mathrm{i}\gamma_0=1/2-\beta_0$, $\mathrm{i}t_0=\log(-\hat{v}e^{i\pi/4}/\sqrt{2})$;

(ii) $p_\nu^*(\cos\hat{t},\sin\hat{t})$, $q_\nu^*(\cos\hat{t},\sin\hat{t})$ $(\nu=1,2)$ *denote* p_ν, q_ν *in Example* 2.1 *with* a_*; *and*

(iii) $c_2=c_2(\gamma_0,a_*,a_1,a_2)$, $c_2'=c_2'(\gamma_0,a_*,a_1,a_2)$ *are polynomials in* (γ_0,a_*,a_1,a_2) *such that* $c_2(\gamma_0,a_*,0,0)=c_2'(\gamma_0,a_*,0,0)=0$.

Remark 2.3. In the case of general parameters θ_0, θ_1, $\theta_\infty\in\mathbb{C}$, as shown above, some first terms of asymptotic series of g_*, β_* may be calculated by the successive substitution method as in the proof of Theorem 2.1 (Section 4.1). In obtaining the higher-order terms, however, integrals such as

$$\int_\infty^x\frac{\cos\xi}{\sin^2\xi}\xi^{-2}d\xi,\qquad \int_\infty^x\cos 2\eta\Big(\int_\infty^\eta\frac{\cos\xi}{\sin^2\xi}\xi^{-2}d\xi\Big)\eta^{-1}d\eta$$

appear, the former being written as

$$-\frac{x^{-2}}{\sin x}-\Big(\log\frac{1-\cos x}{1+\cos x}+4x^3\int_\infty^x\log\frac{1-\cos\xi}{1+\cos\xi}\xi^{-4}d\xi\Big)x^{-3},\quad x^3\int_\infty^x\log\frac{1-\cos\xi}{1+\cos\xi}\xi^{-4}d\xi\ll 1.$$

In treating the general parameter case it seems that another formulation of asymptotic series is necessary.

The expressions of $g(t)$ and $\beta(t)$ are found by successive substitution in Section 4. If the method in [12, §§3 and 4] is applicable to this case, $g(t)$ and $\beta(t)$ may admit convergent series expansions. For example,

Conjecture 2.5. *Suppose that* $\theta_0-\theta_1=\theta_\infty=0$. *Then*

$$\begin{aligned} g(t)&=\phi_0(t)+\sum_{n=1}^\infty\phi_{n+}(t)(t^{-1/2}e^{i\hat{t}})^n+\sum_{n=1}^\infty\phi_{n-}(t)(t^{-1/2}e^{-i\hat{t}})^n, \\ \beta(t)&=\phi_0^*(t)+\sum_{n=1}^\infty\phi_{n+}^*(t)(t^{-1/2}e^{i\hat{t}})^n+\sum_{n=1}^\infty\phi_{n-}^*(t)(t^{-1/2}e^{-i\hat{t}})^n \end{aligned}$$

are convergent for $|t^{-1/2}e^{\pm i\hat{t}}|<r_0$, *where* $\phi_0(t)$, $\phi_0^*(t)$, $\phi_{n\pm}(t)$ *and* $\phi_{n\pm}^*(t)$ *admit asymptotic expansions in powers of* t^{-1} *as* $x\to\infty$ *in a domain containing or close to the positive real axis.*

3 Derivation of (1.6), (1.7)

Multiplying both sides of (1.1) by $2(dy/dt)y^{-1}(y-1)^{-2}$, we write (1.1) in the form

$$\frac{d}{dt}L = -2t^{-1}L - \frac{2t^{-1}y}{(y-1)^2} + 2(1-\theta_0-\theta_1)\frac{t^{-2}}{y-1},$$

where

$$L = \frac{(y_t)^2 - y^2}{y(y-1)^2} + 2(1-\theta_0-\theta_1)\frac{t^{-1}}{y-1} - \frac{t^{-2}}{4}\Big((\theta_0-\theta_1+\theta_\infty)^2 y + (\theta_0-\theta_1-\theta_\infty)^2\frac{1}{y}\Big).$$

On the other hand by [13, (3.13)] set

$$\begin{aligned} a_\phi = A_\phi + \frac{b(t)}{t} &= 1 - 4\frac{(y_t)^2-y^2}{y(y-1)^2} + 4(\theta_0+\theta_1)\frac{t^{-1}(y+1)}{y-1} \\ &+ \frac{t^{-2}(y-1)}{y}((\theta_0-\theta_1+\theta_\infty)^2 y - (\theta_0-\theta_1-\theta_\infty)^2). \end{aligned} \tag{3.1}$$

These functions L and a_ϕ are essentially the Lagrangian functions for (1.1), which is useful in examining properties of the fifth Painlevé transcendents.

Let us set $\psi = (y+1)/(y-1)$. Then the three equations above turn to

$$\begin{aligned} L = \frac{(\psi_t)^2}{\psi^2-1} &- \frac{1}{4}(\psi^2-1) - (1-\theta_0-\theta_1)t^{-1}(1-\psi) \\ &+ \frac{t^{-2}}{4}\Big((\theta_0-\theta_1+\theta_\infty)^2\frac{1+\psi}{1-\psi} + (\theta_0-\theta_1-\theta_\infty)^2\frac{1-\psi}{1+\psi}\Big), \end{aligned} \tag{3.2}$$

$$\frac{d}{dt}L = -2t^{-1}L - \frac{1}{2}t^{-1}(\psi^2-1) + (\theta_0+\theta_1-1)t^{-2}(1-\psi), \tag{3.3}$$

$$\begin{aligned} 4(\psi_t)^2 = (1-\psi^2)(a_\phi-\psi^2) &- 4(\theta_0+\theta_1)t^{-1}\psi(1-\psi^2) \\ &+ 4t^{-2}(2(\theta_0-\theta_1)\theta_\infty\psi + (\theta_0-\theta_1)^2 + \theta_\infty^2). \end{aligned} \tag{3.4}$$

The relation between L and a_ϕ follows from (3.2) and (3.4):

$$L = \frac{1}{4}(1-a_\phi) + (\theta_0+\theta_1-1+\psi)t^{-1} - \frac{1}{2}((\theta_0-\theta_1)^2+\theta_\infty^2)t^{-2}. \tag{3.5}$$

The system consisting of (3.3) and (3.4) with (3.5) is one with respect to ψ and $a_\phi = A_\phi + t^{-1}b(t)$. Then setting $A_\phi = 0$ we obtain (1.6), (1.7).

4 Proofs of the results

Substitution $t/2 = x$, $b(t)/2 - 2\psi(t) = \chi(x)$ changes (1.6) and (1.7) into

$$\begin{aligned} (\psi')^2 = -\psi^2(1-\psi^2) &- (1-\psi^2)(2(\theta_0+\theta_1-1)\psi - \chi)x^{-1} \\ &+ (2(\theta_0-\theta_1)\theta_\infty\psi + (\theta_0-\theta_1)^2 + \theta_\infty^2)x^{-2}, \end{aligned} \tag{4.1}$$

$$\chi' = 2\psi^2 + (2(\theta_0 + \theta_1 - 1)\psi - \chi)x^{-1} \tag{4.2}$$

($\psi' = d\psi/dx$). In (4.1) neglecting the terms with x^{-1}, we have $(\tilde{\psi}')^2 = -\tilde{\psi}^2(1 - \tilde{\psi}^2)$ admitting a general solution

$$\psi_0(x) = \cos^{-1}(x - x_0), \quad (\psi_0')^2 = -\psi_0^2(1 - \psi_0^2),$$

x_0 being an integration constant. In view of (2.1) combined with $y = (1+\psi^{-1})/(1-\psi^{-1})$ let us set

$$\psi = \psi_0(x - x_0 + h(x)) = \cos^{-1}(x - x_0 + h(x))$$

with $h(x) \ll \log x$. Since $\psi = \psi_0(u)|_{u=x-x_0+h(x)}$,

$$\begin{aligned}\psi' &= (1 + h'(x))(\psi_0)_u|_{u=x-x_0+h(x)} = (1 + h'(x))\sqrt{-\psi_0^2(1 - \psi_0^2)}\Big|_{u=x-x_0+h(x)} \\ &= (1 + h'(x))\sqrt{-\psi^2(1 - \psi^2)}.\end{aligned}$$

Hence (4.1) turns to

$$(1 + h')^2 = 1 + F_1(\psi, \chi)x^{-1} + F_2(\psi)x^{-2}$$

with

$$F_1(\psi, \chi) = \frac{2(\theta_0 + \theta_1 - 1)\psi - \chi}{\psi^2}, \quad F_2(\psi) = -\frac{2(\theta_0 - \theta_1)\theta_\infty\psi + (\theta_0 - \theta_1)^2 + \theta_\infty^2}{\psi^2(1 - \psi^2)}, \tag{4.3}$$

which yields

$$h' = \sum_{n=1}^{\infty} G_n(\psi, \chi)x^{-n},$$

$G_n(\psi, \chi)$ being given by

$$\sum_{n=1}^{\infty} G_n(\psi, \chi)x^{-n} = \sum_{m=1}^{\infty} \frac{1}{m!}\binom{1/2}{m}(F_1(\psi, \chi)x^{-1} + F_2(\psi)x^{-2})^m,$$

that is,

$$\begin{aligned}G_1(\psi, \chi) &= \frac{1}{2}F_1(\psi, \chi), \quad G_2(\psi, \chi) = -\frac{1}{8}(F_1(\psi, \chi)^2 - 4F_2(\psi)), \\ G_3(\psi, \chi) &= \frac{1}{16}(F_1(\psi, \chi)^3 - 4F_1(\psi, \chi)F_2(\psi)), \quad \ldots\end{aligned} \tag{4.4}$$

Observing that $\tilde{\chi} = 2\tan(x - x_0)$ solves $\tilde{\chi}' = 2\psi_0^2$, we set

$$\beta = \chi(x) - \chi_0(x) + 4\gamma_0, \quad \chi_0 = 2\tan(x - x_0 + h(x)),$$

γ_0 being an integration constant. Since $\chi_0' = 2(1 + h')\psi^2$, (4.2) is written in the form

$$\beta' = -2h'\psi^2 + (2(\theta_0 + \theta_1 - 1)\psi - \chi_0 + 4\gamma_0 - \beta)x^{-1}$$

$$= -2\psi^2 \sum_{n=2}^{\infty} G_n(\psi, \chi_0 - 4\gamma_0 + \beta)x^{-n}.$$

Finally setting $g = h(x) - \gamma_0 \log x$ we arrive at

$$g' = G_1(\psi, \chi_0 + \beta)x^{-1} + \sum_{n=2}^{\infty} G_n(\psi, \chi_0 - 4\gamma_0 + \beta)x^{-n}.$$

Thus by the substitution

$$\begin{aligned} &t/2 = x, \quad b(t)/2 - 2\psi(t) = \chi(x) = \chi_0(x) - 4\gamma_0 + \beta, \\ &\psi = \cos^{-1}(x - x_0 + \gamma_0 \log x + g), \quad \chi_0(x) = 2\tan(x - x_0 + \gamma_0 \log x + g) \end{aligned}$$

with x_0, t_0 such that $x - x_0 + \gamma_0 \log x = (t - t_0)/2 + \gamma_0 \log t$, we obtain

$$g' = G_1(\psi, \chi_0 + \beta)x^{-1} + \sum_{n=2}^{\infty} G_n(\psi, \chi_0 - 4\gamma_0 + \beta)x^{-n}, \tag{4.5}$$

$$\beta' = -2\psi^2 \sum_{n=2}^{\infty} G_n(\psi, \chi_0 - 4\gamma_0 + \beta)x^{-n} \tag{4.6}$$

with $G_n(\psi, \ldots)$ given by (4.3) and (4.4).

In the main results ψ, β, g are given as functions in t. In what follows these are treated as functions in x, and are denoted by $\psi = \psi(x)$, $\beta = \beta(x)$, $g = g(x)$.

4.1 Proof of Theorem 2.1

If $\theta_0 - \theta_1 = \theta_\infty = 0$, then $F_2(\psi) \equiv 0$ and

$$G_n(\psi, \chi) = c_n F_1(\psi, \chi)^n, \quad c_n = \frac{1}{n!}\binom{1/2}{n}, \quad a = 2(\theta_0 + \theta_1 - 1) = 2(2\theta_0 - 1).$$

Suppose that $g(x) \ll x^{-1}$, $\beta(x) \ll 1$. Note that

$$\cos(\hat{x} + g) = \cos\hat{x} - g\sin\hat{x} - \frac{g^2}{2}\cos\hat{x} + \cdots,$$

where $\hat{x} = x - x_0 + \gamma_0 \log x$. Then (4.5) and (4.6) are written in the form

$$g' = \mathcal{F}(g, \beta, x), \quad \beta' = \mathcal{G}(g, \beta, x). \tag{4.7}$$

Here

$$\begin{aligned} \mathcal{F}(g, \beta, x) =& c_1 \sum_{n=0}^{\infty} \frac{1}{n!} f_{(n)}(\hat{x}, \beta) g^n x^{-1} + \sum_{m=2}^{\infty} c_m \left(\sum_{n=0}^{\infty} \frac{1}{n!} (f^m)_{(n)}(\hat{x}, \beta - 4\gamma_0) g^n \right) x^{-m} \\ =& c_1 \sum_{n=0}^{N} \frac{1}{n!} f_{(n)}(\hat{x}, \beta) g^n x^{-1} + \sum_{m=2}^{N+1} c_m \sum_{n=0}^{N+1-m} \frac{1}{n!} (f^m)_{(n)}(\hat{x}, \beta - 4\gamma_0) g^n x^{-m} \end{aligned}$$

$$
\begin{aligned}
&+O(x^{-N-2})\\
=&\frac{1}{2}\cos\hat{x}(a-2\sin\hat{x}-\beta\cos\hat{x})x^{-1}\\
&+\frac{1}{2}(-a\sin\hat{x}-2\cos 2\hat{x}+\beta\sin 2\hat{x})gx^{-1}\\
&-\frac{1}{8}\cos^2\hat{x}(a-2\sin\hat{x}+(4\gamma_0-\beta)\cos\hat{x})^2x^{-2}+\cdots+O(x^{-N-2}),\\
\mathcal{G}(g,\beta,x)=&-2\sum_{m=2}^{\infty}c_m\left(\sum_{n=0}^{\infty}\frac{1}{n!}(\tilde{f}^m)_{(n)}(\hat{x},\beta-4\gamma_0)g^n\right)x^{-m}\\
=&-2\sum_{m=2}^{N+1}c_m\sum_{n=0}^{N+1-m}\frac{1}{n!}(\tilde{f}^m)_{(n)}(\hat{x},\beta-4\gamma_0)g^nx^{-m}+O(x^{-N-2})\\
=&\frac{1}{4}(a-2\sin\hat{x}+(4\gamma_0-\beta)\cos\hat{x})^2x^{-2}+\cdots+O(x^{-N-2})
\end{aligned}
$$

with

$$
\begin{aligned}
&f(u,\beta)=\cos u(a-2\sin u-\beta\cos u)=a\cos u-\sin 2u-\beta\cos^2 u,\\
&f^m(u,\beta)=f(u,\beta)^m,\qquad \tilde{f}_m(u,\beta)=\cos^{-2}u\,f^m(u,\beta),\\
&f_{(n)}(u,\beta)=(\partial/\partial u)^n f(u,\beta),
\end{aligned}
$$

in $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$, in which $\cos\hat{x}$, $\sin\hat{x}\ll 1$. Furthermore, for $g_{(j)}$, $\beta_{(j)}\ll x^{-1}$ $(j=1,2)$,

$$
\begin{aligned}
&|\mathcal{F}(g_{(2)},\beta_{(2)},x)-\mathcal{F}(g_{(1)},\beta_{(1)},x)|\ll|x|^{-1}(|g_{(2)}-g_{(1)}|+|\beta_{(2)}-\beta_{(1)}|),\\
&|\mathcal{G}(g_{(2)},\beta_{(2)},x)-\mathcal{G}(g_{(1)},\beta_{(1)},x)|\ll|x|^{-2}(|g_{(2)}-g_{(1)}|+|\beta_{(2)}-\beta_{(1)}|).
\end{aligned}
$$

Note that $\mathcal{G}(g,\beta,x)\ll x^{-2}$, which implies $\beta(x)\ll x^{-1}$ as $x\to\infty$. Then (g,β) satisfies the system of integral equations

$$
g(x)=\int_\infty^x\mathcal{F}(g(\xi),\beta(\xi),\xi)d\xi,\quad \beta(x)=\int_\infty^x\mathcal{G}(g(\xi),\beta(\xi),\xi)d\xi, \tag{4.8}
$$

where the contour is a path tending to ∞ through $\Sigma_0(t_\infty,R_0,\gamma_0,\delta_0)$. Let (g_m,β_m) be the sequence given by the successive substitution:

$$
\begin{aligned}
&g_0(x)=0,\quad \beta_0(x)=0,\\
&g_{m+1}(x)=\int_\infty^x\mathcal{F}(g_m(\xi),\beta_m(\xi),\xi)d\xi,\quad \beta_{m+1}(x)=\int_\infty^x\mathcal{G}(g_m(\xi),\beta_m(\xi),\xi)d\xi.
\end{aligned}
$$

Now let us observe the case, say $N=2$. Then

$$
\begin{aligned}
&\mathcal{F}(g,\beta,x)=G_1x^{-1}+G_2x^{-2}+G_3x^{-3}+O(x^{-4}),\\
&\mathcal{G}(g,\beta,x)=-2\psi^2(G_2x^{-2}+G_3x^{-3})+O(x^{-4}),
\end{aligned}
$$

where

$$\begin{aligned}
G_1 =& -\frac{\beta}{4} + \frac{a}{2}\cos\hat{x} - \frac{1}{2}\sin 2\hat{x} - \frac{\beta}{4}\cos 2\hat{x} + \Big(-\frac{a}{2}\sin\hat{x} - \cos 2\hat{x} + \frac{\beta}{2}\sin 2\hat{x}\Big)g \\
&+ \Big(-\frac{a}{2}\cos\hat{x} + 2\sin 2\hat{x} + \beta\cos 2\hat{x}\Big)\frac{g^2}{2} + \cdots\,, \\
G_2 =& -\frac{1}{16}\Big(a^2 + \frac{3}{4}\tilde{\beta}^2 + 1 + \frac{1}{4}(\tilde{\beta}^2 - 4)\cos 4\hat{x} + \tilde{\beta}\sin 4\hat{x} - a\tilde{\beta}\cos 3\hat{x} - 2a\sin 3\hat{x} \\
&+ (a^2 + \tilde{\beta}^2)\cos 2\hat{x} + 2\tilde{\beta}\sin 2\hat{x} - 3a\tilde{\beta}\cos\hat{x} - 2a\sin\hat{x}\Big) \\
&- \frac{1}{16}\Big(-(\tilde{\beta}^2 - 4)\sin 4\hat{x} + 4\tilde{\beta}\cos 4\hat{x} + 3a\tilde{\beta}\sin 3\hat{x} - 6a\cos 3\hat{x} \\
&- 2(a^2 + \tilde{\beta}^2)\sin 2\hat{x} + 4\tilde{\beta}\cos 2\hat{x} + 3a\tilde{\beta}\sin\hat{x} - 2a\cos\hat{x}\Big)g + \cdots\,, \\
G_3 =& -\frac{1}{256}\tilde{\beta}(5\tilde{\beta}^2 + 18a^2 + 12) + (\cdots)\cos 6\hat{x} + \cdots\,, \\
-2\psi^2 G_2 =& \frac{1}{4}\Big(a^2 + \frac{\tilde{\beta}^2}{2} + 2 + \frac{1}{2}(\tilde{\beta}^2 - 4)\cos 2\hat{x} + 2\tilde{\beta}\sin 2\hat{x} - 2a\tilde{\beta}\cos\hat{x} - 4a\sin\hat{x}\Big) \\
&+ \frac{1}{4}\Big(-(\tilde{\beta}^2 - 4)\sin 2\hat{x} + 4\tilde{\beta}\cos 2\hat{x} + 2a\tilde{\beta}\sin\hat{x} - 4a\cos\hat{x}\Big)g + \cdots\,, \\
-2\psi^2 G_3 =& \frac{3}{64}\tilde{\beta}(\tilde{\beta}^2 + 4a^2 + 4) + (\cdots)\cos 4\hat{x} + \cdots
\end{aligned}$$

with $\tilde{\beta} = \beta - 4\gamma_0$.

Note the fact that every finite product $\Pi\cos m_i x\,\sin n_j x$ with $m_i,\, n_j \in \mathbb{Z}$ is expressed by a linear combination of $\cos mx$, $\sin nx$, $m,\, n \in \mathbb{Z}$ over $\mathbb{Q}$. The following lemma on asymptotic expansions of integrals is verified by repeating integration by parts:

Lemma 4.1. *For a given positive integers $m,\, k \geq 1$,*

$$\begin{aligned}
\int_{\infty}^{x} \xi^{-k}\cos m\xi\, d\xi &\sim \sin mx \sum_{n=0}^{\infty} c_{2n}(m,k)x^{-k-2n} + \cos mx \sum_{n=0}^{\infty} c_{2n+1}(m,k)x^{-k-2n-1}, \\
\int_{\infty}^{x} \xi^{-k}\sin m\xi\, d\xi &\sim \cos mx \sum_{n=0}^{\infty} \tilde{c}_{2n}(m,k)x^{-k-2n} + \sin mx \sum_{n=0}^{\infty} \tilde{c}_{2n+1}(m,k)x^{-k-2n-1}
\end{aligned}$$

with $c_n(m,k)$, $\tilde{c}_n(m,k) \in \mathbb{C}$ as $x \to \infty$ through the strip $\operatorname{Re} x > t_\infty$, $|\operatorname{Im} x| < R_0$.

Careful computation by using integration by parts leads us to

$$\begin{aligned}
g_1 =& \int_{\infty}^{x} \mathcal{F}(0,0,\xi)d\xi = \frac{1}{2}\int_{\infty}^{x}(a\cos\hat{\xi} - \sin 2\hat{\xi})\xi^{-1}d\xi \\
&- \frac{1}{16}\int_{\infty}^{x}(a^2 + 12\gamma_0^2 + 1 + (4\gamma_0^2 - 1)\cos 4\hat{\xi} - 4\gamma_0\sin 4\hat{\xi} - \cdots)\xi^{-2}d\xi \\
&- \frac{\gamma_0}{64}(40\gamma_0^2 + 9a^2 + 6)x^{-2} + O(x^{-3}) \\
=& \frac{1}{2}\int_{\infty}^{x}\Big(a(\sin\hat{\xi})' + \frac{1}{2}(\cos 2\hat{\xi})'\Big)(\hat{\xi}')^{-1}\xi^{-1}d\xi + \frac{1}{16}(a^2 + 12\gamma_0^2 + 1)x^{-1} \\
&- \frac{1}{64}((4\gamma_0^2 - 1)\sin 4\hat{x} + 4\gamma_0\cos 4\hat{x} + \cdots + \gamma_0(40\gamma_0^2 + 9a^2 + 6))x^{-2} + O(x^{-3})
\end{aligned}$$

$$=\Big(\frac{1}{4}\cos 2\hat{x}+\frac{a}{2}\sin\hat{x}+\frac{1}{16}(12\gamma_0^2+a^2+1)\Big)x^{-1}+(g_1)^{(2)}x^{-2}+O(x^{-3})$$

with

$$\begin{aligned}(g_1)^{(2)}=&-\frac{1}{16}\Big(\gamma_0\cos 4\hat{x}+\frac{1}{4}(4\gamma_0^2-1)\sin 4\hat{x}+\frac{2}{3}a\cos 3\hat{x}+\frac{4}{3}a\gamma_0\sin 3\hat{x}+8\gamma_0\cos 2\hat{x}\\ &+\frac{1}{2}(a^2+16\gamma_0^2-4)\sin 2\hat{x}+10a\cos\hat{x}+20a\gamma_0\sin\hat{x}+\frac{\gamma_0}{4}(40\gamma_0^2+9a^2+6)\Big),\end{aligned}$$

and

$$\begin{aligned}\beta_1=&\int_\infty^x\mathcal{G}(0,0,\xi)d\xi=\frac{1}{4}\int_\infty^x((8\gamma_0^2+a^2+2)+(8\gamma_0^2-2)\cos 2\hat{\xi}-8\gamma_0\sin 2\hat{\xi}-\cdots)\xi^{-2}d\xi\\ &+\frac{3}{8}\gamma_0(4\gamma_0^2+a^2+1)x^{-2}+O(x^{-3})\\ =&-\frac{1}{4}(8\gamma_0^2+a^2+2)x^{-1}+(\beta_1)^{(2)}x^{-2}+O(x^{-3})\end{aligned}$$

with

$$(\beta_1)^{(2)}=\gamma_0\cos 2\hat{x}+\frac{1}{4}(4\gamma_0^2-1)\sin 2\hat{x}+a\cos\hat{x}+2a\gamma_0\sin\hat{x}+\frac{3}{8}\gamma_0(4\gamma_0^2+a^2+1).$$

Furthermore,

$$\begin{aligned}g_2-g_1=&\int_\infty^x(\mathcal{F}(g_1,\beta_1,\xi)-\mathcal{F}(0,0,\xi))d\xi\\ =&-\frac{1}{4}\int_\infty^x\Big((1+\cos 2\hat{\xi})\beta_1+2(a\sin\hat{\xi}+2\cos 2\hat{\xi}-\beta_1\sin 2\hat{\xi})g_1\Big)\xi^{-1}d\xi\\ &-\frac{1}{4}\int_\infty^x(a\cos\hat{\xi}-4\sin 2\hat{\xi}-2\beta_1\cos 2\hat{\xi})g_1^2\xi^{-1}d\xi\\ &-\frac{1}{16}\int_\infty^x(-6\gamma_0-2\gamma_0\cos 4\hat{\xi}+\sin 4\hat{\xi}-\cdots)\beta_1\xi^{-2}d\xi\\ &-\frac{1}{4}\int_\infty^x((1-4\gamma_0^2)\sin 4\hat{\xi}-4\gamma_0\cos 4\hat{\xi}+\cdots)g_1\xi^{-2}d\xi+O(x^{-3})\\ =&\frac{1}{16}(-8\gamma_0^2+a^2)x^{-1}+(g_2-g_1)^{(2)}x^{-2}+O(x^{-3})\end{aligned}$$

with

$$\begin{aligned}(g_2-g_1)^{(2)}=&\frac{1}{32}\sin 4\hat{x}-\frac{5}{48}a\cos 3\hat{x}+\frac{1}{32}(20\gamma_0^2+3)\sin 2\hat{x}\\ &+\frac{a}{32}(12\gamma_0^2+a^2-9)\cos\hat{x}+\frac{\gamma_0}{64}(36\gamma_0^2-10a^2+1),\end{aligned}$$

and

$$\begin{aligned}\beta_2-\beta_1=&\int_\infty^x(\mathcal{G}(g_1,\beta_1,\xi)-\mathcal{G}(0,0,\xi))d\xi\\ =&\frac{1}{4}\int_\infty^x\Big(\frac{1}{2}(\beta_1-8\gamma_0)(1+\cos 2\hat{\xi})+2(\sin 2\hat{\xi}-a\cos\hat{\xi})\Big)\beta_1\xi^{-2}d\xi\end{aligned}$$

$$\begin{aligned}
&\quad -\int_{\infty}^{x}((4\gamma_0^2-1)\sin 2\hat{\xi}+4\gamma_0\cos 2\hat{\xi}+2a\gamma_0\sin\hat{\xi}+a\cos\hat{\xi})g_1\xi^{-2}d\xi+O(x^{-3})\\
&= -\frac{\gamma_0}{8}(8\gamma_0^2-a^2)x^{-2}+O(x^{-3}),\\
g_3-g_2 &= \int_{\infty}^{x}(\mathcal{F}(g_2,\beta_2,\xi)-\mathcal{F}(g_1,\beta_1,\xi))d\xi\\
&= -\int_{\infty}^{x}\Big(\frac{1}{2}(a\sin\hat{\xi}+2\cos 2\hat{\xi})(g_2-g_1)+\frac{1}{4}(1+\cos 2\hat{\xi})(\beta_2-\beta_1)\Big)\xi^{-1}d\xi+O(x^{-3})\\
&= -\frac{1}{64}(8\gamma_0^2-a^2)(\gamma_0+2a\cos\hat{x}-2\sin 2\hat{x})x^{-2}+O(x^{-3}),\\
\beta_3-\beta_2 &= \int_{\infty}^{x}(\mathcal{G}(g_2,\beta_2,\xi)-\mathcal{G}(g_1,\beta_1,\xi))d\xi\\
&= \int_{\infty}^{x}((4\gamma_0^2-1)\sin 2\hat{\xi}+4\gamma_0\cos 2\hat{\xi}+2a\gamma_0\sin\hat{\xi}+a\cos\hat{\xi})(g_2-g_1)\xi^{-2}d\xi\\
&\quad +\int_{\infty}^{x}(\cdots)(\beta_2-\beta_1)\xi^{-2}d\xi+O(x^{-3})\ll x^{-3},\\
g_4-g_3 &= \int_{\infty}^{x}((\cdots)(g_3-g_2)+(\cdots)(\beta_3-\beta_2))\xi^{-2}d\xi+O(x^{-3})\ll x^{-3}.
\end{aligned}$$

Thus we obtain the result in Example 2.1, and, p_ν, q_ν are determined for $\nu\le 2$.

In the case of a given large N, the leading terms of $g_{m+1}-g_m$, $\beta_{m+1}-\beta_m$ are as follows:

$$\begin{aligned}
g_{m+1}-g_m &= g_*^{(m)}(\cos\hat{x},\sin\hat{x})x^{-[m/2]-2+\epsilon(m)}+O(x^{-[m/2]-3+\epsilon(m)}),\\
\beta_{m+1}-\beta_m &= \beta_*^{(m)}(\cos\hat{x},\sin\hat{x})x^{-[m/2]-2}+O(x^{-[m/2]-3})
\end{aligned}\tag{4.9}$$

for $m\ge 2$. Here $\epsilon(m)=(1+(-1)^m)/2$, $[t]$ denotes the greatest integer not exceeding t, and $g_*^{(m)}$ is linear in 1, $\cos\hat{x}$, $\sin 2\hat{x}$, $\cos(2k-1)\hat{x}$, $\sin 2k\hat{x}$ $(k\ge 1)$. As shown in the computation above (4.9) is valid for $m=2$, and for $m\ge 3$ (4.9) is verified by induction on m by using the fact: the leading terms of $g_{m+1}-g_m$ and $\beta_{m+1}-\beta_m$ result from

$$\begin{aligned}
&\int_{\infty}^{x}(\mathcal{F}(g_m,\beta_m,\xi)-\mathcal{F}(g_{m-1},\beta_{m-1},\xi))d\xi\\
&\qquad\qquad = \int_{\infty}^{x}(\mathcal{F}^*_{m,1}(g_m-g_{m-1})+\mathcal{F}^*_{m,2}(\beta_m-\beta_{m-1}))\xi^{-1}d\xi,\\
&\int_{\infty}^{x}(\mathcal{G}(g_m,\beta_m,\xi)-\mathcal{G}(g_{m-1},\beta_{m-1},\xi))d\xi\\
&\qquad\qquad = \int_{\infty}^{x}(\mathcal{G}^*_{m,1}(g_m-g_{m-1})+\mathcal{G}^*_{m,2}(\beta_m-\beta_{m-1}))\xi^{-2}d\xi
\end{aligned}$$

with $\mathcal{F}^*_{m,j}=\mathcal{F}^*_{m,j}(g_m,g_{m-1},\beta_m,\beta_{m-1},x)$, $\mathcal{G}^*_{m,j}=\mathcal{G}^*_{m,j}(g_m,g_{m-1},\beta_m,\beta_{m-1},x)$ such that

$$\begin{aligned}
\mathcal{F}^*_{m,1} &= -\frac{a}{2}\sin\hat{\xi}-\cos 2\hat{\xi}+(\cdots),\\
\mathcal{F}^*_{m,2} &= -\frac{1}{4}(1+\cos 2\hat{\xi})+(\cdots),\\
\mathcal{G}^*_{m,1} &= -(4\gamma_0^2-1)\sin 2\hat{\xi}-4\gamma_0\cos 2\hat{\xi}-2a\gamma_0\sin\hat{\xi}-a\cos\hat{\xi}+(\cdots),\\
\mathcal{G}^*_{m,2} &= -\gamma_0(1+\cos 2\hat{\xi})+\frac{1}{2}(\sin 2\hat{\xi}-a\cos\hat{\xi})+(\cdots)
\end{aligned}\tag{4.10}$$

with $(\cdots) \ll |\xi|^{-1} + |g_m| + |g_{m-1}| + |\beta_m| + |\beta_{m-1}|$. Thus for each ν by the finite-times repetition of the substitution procedure, by Lemma 4.1 we get the asymptotic series g_m, β_m $(m \le 2\nu)$, and p_ν, q_ν.

In $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$ we have, as long as $|g_m|$, $|\beta_m| \le 4K_0|x|^{-1}$, for $m \ge m_0$

$$|g_{m_0+1} - g_{m_0}| + |\beta_{m_0+1} - \beta_{m_0}| \le M_0|x|^{-m_0/2},$$
$$|g_{m+1} - g_m| + |\beta_{m+1} - \beta_m| \le M_1 \int_\infty^x (|g_m - g_{m-1}| + |\beta_m - \beta_{m-1}|)|\xi|^{-1}|d\xi|,$$

where m_0 is a positive number fixed later, $M_0 > 0$ depends on m_0, $M_1 > 0$ does not depend on m_0, and $K_0 > 0$ is such that $|\beta_1| + |g_1| \le K_0|x|^{-1}$. Then we may inductively show that

$$|g_{m+1} - g_m| + |\beta_{m+1} - \beta_m| \le (2M_1/m_0)^{m-m_0} M_0|x|^{-m_0/2}$$

for $m \ge m_0$ in $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$. Choose $m_0 > 4M_1$, and again choose t_∞ sufficiently large in such a way that

$$|g_1| + |\beta_1| + \sum_{m=1}^{m_0-1} (|g_{m+1} - g_m| + |\beta_{m+1} - \beta_m|) \le 2K_0|x|^{-1},$$
$$\sum_{m=m_0}^{\infty} (|g_{m+1} - g_m| + |\beta_{m+1} - \beta_m|) \le 2M_0|x|^{-m_0/2} \le K_0|x|^{-1}$$

in $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$. Then $g_\infty = g_1 + \sum_{m=1}^\infty (g_{m+1} - g_m)$ and $\beta_\infty = \beta_1 + \sum_{m=1}^\infty (\beta_{m+1} - \beta_m)$ converge absolutely and uniformly in $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$, and (g_∞, β_∞) solves (4.8), which implies the existence of a family of solutions of system (1.6), (1.7).

For any integer N,

$$\Big|\sum_{\nu=1}^{N} p_\nu\,(2x)^{-\nu} - g_1 - \sum_{m=1}^{2N+1} (g_{m+1} - g_m)\Big| + \Big|\sum_{\nu=1}^{N} q_\nu\,(2x)^{-\nu} - \beta_1 - \sum_{m=1}^{2N+1} (\beta_{m+1} - \beta_m)\Big| \ll |x|^{-N-1}. \tag{4.11}$$

Choose again m_0 together with t_∞ such that $m_0 \ge 2(N+1)$ if necessary. Then (4.11) combined with

$$\sum_{m=2(N+1)}^{\infty} (|g_{m+1} - g_m| + |\beta_{m+1} - \beta_m|) = \sum_{m=2(N+1)}^{m_0} + \sum_{m=m_0+1}^{\infty} \ll |x|^{-N-1} + |x|^{-m_0/2} \ll |x|^{-N-1}$$

yields

$$g_\infty = \sum_{\nu=1}^{N} p_\nu(\cos\hat{x}, \sin\hat{x})(2x)^{-\nu} + O(x^{-N-1}),$$
$$\beta_\infty = \sum_{\nu=1}^{N} q_\nu(\cos\hat{x}, \sin\hat{x})(2x)^{-\nu} + O(x^{-N-1}).$$

This completes the proof of Theorem 2.1.

4.2 Proof of Theorem 2.3

As long as $t = 2x$ is in $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$ the solution (y_*, z_*) may be written in the form

$$\begin{aligned}
&y_*(t) = \frac{1+\cos u}{1-\cos u}, \quad t^{-1}z_*(t) = \frac{\sin u}{8(1-\sin u)} + O(t^{-1}), \qquad (4.12)\\
&u = (t-t_0)/2 + \gamma_0 \log t + g_*(t), \quad g_*(t) \ll t^{-1},
\end{aligned}$$

where $\mathrm{i}\gamma_0 = 1/2 - \beta_0$ and $\mathrm{i}t_0 = \log(-\hat{v}e^{\pi\mathrm{i}/4}/\sqrt{2})$, and by (1.3)

$$t^{-1}z_* = -\frac{(y_*)_t - y_*}{2(y_*-1)^2} + O(t^{-1}) = -\frac{(y_*)_u/2 - y_*}{2(y_*-1)^2} - \frac{(y_*)_u(g_*)_t}{2(y_*-1)^2} + O(t^{-1}),$$

since $(y_*)_t = (1/2 + O(t^{-1}) + (g_*)_t)(y_*)_u$. By (4.12), $t^{-1}z_* = -((y_*)_u/2 - y_*)(y_*-1)^{-2}/2 + O(t^{-1})$, and hence $(g_*)_t \ll t^{-1}$. Combining this with (4.5), in which $F_1(\psi_*, \ldots) = (\psi_*)^{-2}(2(\theta_0+\theta_1-1)\psi_* - \chi_0 - \beta_*)$, we have $\beta_*(x) \ll 1$. Furthermore from $\beta'_* = \mathcal{G}(g_*, \beta_*, x) \ll x^{-2}$, it follows that $\beta_*(x) - c_0 \ll x^{-1}$ as $x \to \infty$. Since γ_0 is an integration constant we may set $c_0 = 0$. Thus we have

Lemma 4.2. *$g_*(x), g'_*(x) \ll x^{-1}$, $\beta_*(x) \ll x^{-1}$ as $x \to \infty$ through $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$.*

By (4.12), (g_*, β_*) fulfills (4.7), and by Lemma 4.2

$$g_* = \int_\infty^x \mathcal{F}(g_*, \beta_*, \xi)d\xi, \quad \beta_* = \int_\infty^x \mathcal{G}(g_*, \beta_*, \xi)d\xi.$$

Then by the successive substitution method as in Section 4.1, it is shown that (g_*, β_*) admits the same asymptotic representation as of (g_∞, β_∞):

$$(g_\infty, \beta_\infty) = \sum_{\nu=1}^{N} (p_\nu(\cos\hat{x}, \sin\hat{x}), q_\nu(\cos\hat{x}, \sin\hat{x}))(2x)^{-\nu} + O(x^{-N-1}),$$

since as long as $g_*, \beta_* \ll x^{-1}$ the substitution procedure is possible. Hence for sufficiently large m_0,

$$\begin{aligned}
&|g_* - g_\infty| + |\beta_* - \beta_\infty| \ll |x|^{-m_0},\\
&|g_* - g_\infty| + |\beta_* - \beta_\infty|\\
&\qquad \le \int_\infty^x (|\mathcal{F}(g_*, \beta_*, \xi) - \mathcal{F}(g_\infty, \beta_\infty, \xi)| + |\mathcal{G}(g_*, \beta_*, \xi) - \mathcal{G}(g_\infty, \beta_\infty, \xi)|)|d\xi|\\
&\qquad \le M_1^* \int_\infty^x (|g_* - g_\infty| + |\beta_* - \beta_\infty|)|\xi|^{-1}|d\xi|,
\end{aligned}$$

where M_1^* is independent of m_0. Take $m_0 > 2M_1^*$. Then $|g_* - g_\infty| + |\beta_* - \beta_\infty| \ll 2^{-(m-m_0)}|x|^{-m_0}$ for every $m \ge m_0$. This implies $g_* - g_\infty \equiv \beta_* - \beta_\infty \equiv 0$, which completes the proof of Theorem 2.3.

4.3 Proof of Theorem 2.4

In the general case $F_2(\psi) \not\equiv 0$ is added in G_n (cf. (4.4)). The system turns to

$$\begin{aligned} g' &= \frac{1}{2}F_1(\psi, \chi_0 + \beta)x^{-1} - \frac{1}{8}(F_1(\psi, \chi_0 - 4\gamma_0 + \beta)^2 - 4F_2(\psi))x^{-2} + O(x^{-3}), \\ \beta' &= \frac{\psi^2}{4}(F_1(\psi, \chi_0 - 4\gamma_0 + \beta)^2 - 4F_2(\psi))x^{-2} + O(x^{-3}), \end{aligned}$$

where

$$F_2(\psi) = \frac{\cos^3(\hat{x} + g)}{\sin^2(\hat{x} + g)}(a_1 + a_2\cos(\hat{x} + g)), \quad a_1 = 2(\theta_0 - \theta_1)\theta_\infty, \quad a_2 = (\theta_0 - \theta_1)^2 + \theta_\infty^2$$

is also bounded in $\Sigma_0(t_\infty, R_0, \gamma_0, \delta_0)$. Put $a_* = 2(\theta_0 + \theta_1 - 1)$ in place of $a = 2(2\theta_0 - 1)$ in Section 4.1. Since Lemma 4.2 is also valid in the general case, $(g_*(t), \beta_*(t))$ satisfies

$$g_* = \int_\infty^x \mathcal{F}(g_*, \beta_*, \xi)d\xi, \quad \beta_* = \int_\infty^x \mathcal{G}(g_*, \beta_*, \xi)d\xi$$

with $F_2(\psi) \not\equiv 0$, where $\mathcal{F}$ and $\mathcal{G}$ are written in the form

$$\begin{aligned} &\mathcal{F}(g_*, \beta_*, \xi) \\ =&\frac{1}{2}(a_*\cos\hat{\xi} - \sin 2\hat{\xi})\xi^{-1} - \frac{1}{4}(1 + \cos 2\hat{\xi})\beta_*\xi^{-1} - \frac{1}{2}(a_*\sin\hat{\xi} + 2\cos 2\hat{\xi})g_*\xi^{-1} \\ &- \frac{1}{16}\Big((4\gamma_0^2 - 1)\cos 4\hat{\xi} - 4\gamma_0\sin 4\hat{\xi} + 4a_*\gamma_0\cos 3\hat{\xi} - 2a_*\sin 3\hat{\xi} \\ &+ (16\gamma_0^2 + a_*^2)\cos 2\hat{\xi} - 8\gamma_0\sin 2\hat{\xi} + 12a_*\gamma_0\cos\hat{\xi} - 2a_*\sin\hat{\xi} + 12\gamma_0^2 + a_*^2 + 1\Big)\xi^{-2} \\ &+ \frac{1}{2}\frac{\cos^3\hat{\xi}}{\sin^2\hat{\xi}}(a_1 + a_2\cos\hat{\xi})\xi^{-2} + \frac{1}{2}\Big(\frac{\cos^3 u}{\sin^2 u}(a_1 + a_2\cos u)\Big)_u\Big|_{u=\hat{\xi}} g_*\xi^{-2} + O(\xi^{-3}), \\ &\mathcal{G}(g_*, \beta_*, \xi) \\ =&\frac{1}{4}\Big((8\gamma_0^2 - 2)\cos 2\hat{\xi} - 4\gamma_0\sin 2\hat{\xi} + 8a_*\gamma_0\cos\hat{\xi} - 4a_*\sin\hat{\xi} + (8\gamma_0^2 + a_*^2 + 2)\Big)\xi^{-2} \\ &- ((4\gamma_0^2 - 1)\sin 2\hat{\xi} + 4\gamma_0\cos 2\hat{\xi} + 2a_*\gamma_0\sin\hat{\xi} + a_*\cos\hat{\xi})g_*\xi^{-2} \\ &- \frac{1}{2}(2\gamma_0(1 + \cos 2\hat{\xi}) - (\sin 2\hat{\xi} - a_*\cos\hat{\xi}))\beta_*\xi^{-2} \\ &- \frac{\cos\hat{\xi}}{\sin^2\hat{\xi}}(a_1 + a_2\cos\hat{\xi})\xi^{-2} - \Big(\frac{\cos u}{\sin^2 u}(a_1 + a_2\cos u)\Big)_u\Big|_{u=\hat{\xi}} g_*\xi^{-2} + O(\xi^{-3}). \end{aligned}$$

The asymptotic expression as in the theorem is calculated by successive substitution. In the desired expression the polynomials in $(\cos\hat{x}, \sin\hat{x})$ resulting from the parts other than $F_2(\psi)$ are the same as in Section 4.1 replaced a by a_*. Let us calculate the contribution from $F_2(\psi)$, which we denote by (g_m^*, β_m^*). Then

$$g_1^* = \frac{1}{2}\int_\infty^x \frac{\cos^3\hat{\xi}}{\sin^2\hat{\xi}}(a_1 + a_2\cos\hat{\xi})\xi^{-2}d\xi + \cdots$$

$$
\begin{aligned}
&=\frac{a_1}{2}\int_\infty^x \cos\hat{\xi}\Big(\frac{1}{\sin^2\hat{\xi}}-1\Big)\xi^{-2}d\xi+\frac{a_2}{2}\int_\infty^x\Big(\frac{1}{\sin^2\hat{\xi}}-2+\sin^2\hat{\xi}\Big)\xi^{-2}d\xi+\cdots\\
&=\frac{3}{4}a_2x^{-1}-\frac{a_1}{2}\Big(\frac{1}{\sin\hat{x}}+\sin\hat{x}\Big)x^{-2}-\frac{a_2}{2}\Big(\frac{\cos\hat{x}}{\sin\hat{x}}+\frac{1}{4}\sin 2\hat{x}\Big)x^{-2}+c_3x^{-2}+O(x^{-3}),
\end{aligned}
$$

$$
\begin{aligned}
\beta_1^* &=-\int_\infty^x\frac{\cos\hat{\xi}}{\sin^2\hat{\xi}}(a_1+a_2\cos\hat{\xi})\xi^{-2}d\xi+\cdots\\
&=-\,a_1\int_\infty^x\frac{\cos\hat{\xi}}{\sin^2\hat{\xi}}\xi^{-2}d\xi-a_2\int_\infty^x\Big(\frac{1}{\sin^2\hat{\xi}}-1\Big)\xi^{-2}d\xi+\cdots\\
&=-\,a_2x^{-1}+\frac{1}{4}\Big(2\gamma_0\cos 2\hat{x}+(4\gamma_0^2-1)\sin 2\hat{x}+4a_*\cos\hat{x}+8a_*\gamma_0\sin\hat{x}\Big)x^{-2}\\
&\qquad+\frac{a_1}{\sin\hat{x}}x^{-2}+\frac{a_2\cos\hat{x}}{\sin\hat{x}}x^{-2}+c_3'x^{-2}+O(x^{-3}),
\end{aligned}
$$

and

$$
\begin{aligned}
g_2^*-g_1^* &=-\frac{1}{4}\int_\infty^x(1+\cos 2\hat{\xi})\beta_1^*\xi^{-1}d\xi-\frac{1}{2}\int_\infty^x(a_*\sin\hat{\xi}+2\cos 2\hat{\xi})g_1^*\xi^{-1}d\xi\\
&\qquad+\frac{1}{2}\int_\infty^x\Big(\frac{\cos^3 u}{\sin^2 u}(a_1+a_2\cos u)\Big)_u\Big|_{u=\hat{\xi}}g_1^*\xi^{-2}d\xi+\cdots\\
&=-\frac{a_2}{4}x^{-1}+\frac{a_2}{8}(3a_*\cos\hat{x}-2\sin 2\hat{x})x^{-2}+c_3''x^{-2}+O(x^{-3}),\\
\beta_2^*-\beta_1^* &=-\frac{1}{2}\int_\infty^x(2\gamma_0(1+\cos 2\hat{\xi})-(\sin 2\hat{\xi}-a\cos\hat{\xi}))\beta_1\xi^{-2}d\xi\\
&\qquad-\int_\infty^x((4\gamma_0^2-1)\sin 2\hat{\xi}+4\gamma_0\cos 2\hat{\xi}+2a_*\gamma_0\sin\hat{\xi}+a_*\cos\hat{\xi})g_1\xi^{-2}d\xi\\
&\qquad-\int_\infty^x\Big(\frac{\cos u}{\sin^2 u}(a_1+a_2\cos u)\Big)_u\Big|_{u=\hat{\xi}}g_1^*\xi^{-2}d\xi+\cdots=c_3^{(3)}x^{-2}+O(x^{-3}),\\
g_3^*-g_2^* &=-\frac{1}{4}\int_\infty^x(1+\cos 2\hat{\xi})(\beta_2^*-\beta_1^*)\xi^{-1}d\xi-\frac{1}{2}\int_\infty^x(a_*\sin\hat{\xi}+2\cos 2\hat{\xi})(g_2^*-g_1^*)\xi^{-1}d\xi\\
&\qquad+\frac{1}{2}\int_\infty^x\Big(\frac{\cos^3 u}{\sin^2 u}(a_1+a_2\cos u)\Big)_u\Big|_{u=\hat{\xi}}(g_2^*-g_1^*)\xi^{-2}d\xi+\cdots\\
&=-\frac{a_2}{8}(a_*\cos\hat{x}-\sin 2\hat{x})x^{-2}+c_3^{(4)}x^{-2}+O(x^{-3}),\\
\beta_3^*-\beta_2^* &\ll x^{-3},\\
g_4^*-g_3^* &\ll x^{-3},\quad \beta_4^*-\beta_3^*\ll x^{-3},\quad \ldots,
\end{aligned}
$$

which yields the terms of order t^{-2} as in theorem. Thus Theorem 2.4 is obtained.

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